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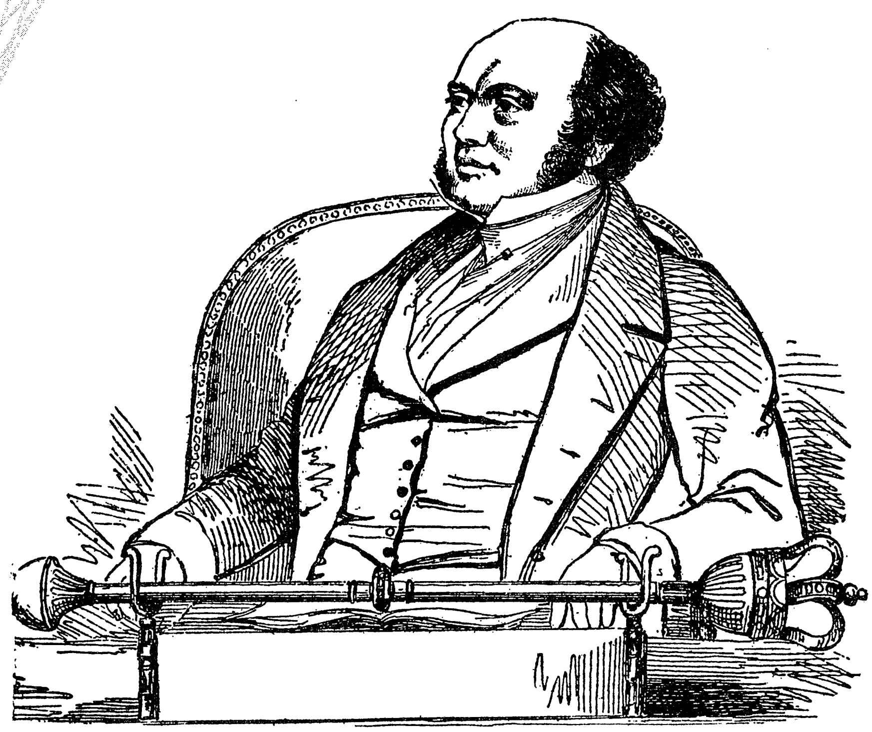

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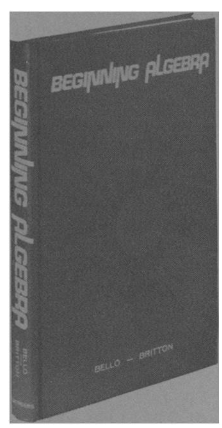

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COVER: A drawing of Sir William Rowan Hamilton, the discoverer of quaternions and of the Hamiltonian operator of physics. This likeness, one of a few in existence, was done by an unknown artist and shows Hamilton with his mace of office as Astronomer Royal.

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Mathematics Magazine is a journal of collegiate mathematics designed to enrich undergraduate study of the mathematical sciences. The *Magazine* should be an inviting, informal journal emphasizing good mathematical exposition of interest to undergraduate students. Manuscripts accepted for publication in the *Magazine* should be written in a clear and lively expository style. The *Magazine* is not a research journal, so papers written in the terse "theorem-proof-corollary-remark" style will ordinarily be unsuitable for publication. Articles printed in the *Magazine* should be of a quality and level that makes it realistic for teachers to use them to supplement their regular courses. The editors especially invite manuscripts that provide insight into applications and history of mathematics. We welcome other informal contributions, for example, brief notes, mathematical games, graphics and humor.

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ABOUT OUR AUTHORS

Bartel Leendert van der Waerden ("Hamilton's Discovery of Quaternions") was born in 1903, and received his doctorate from the University of Amsterdam in 1926. He served as a professor at several major universities in the Netherlands and Germany, and for many years has been a professor at the University of Zurich. His *Modern Algebra* in two volumes (1930 and 1931, published originally in German but translated into several languages) is the classic introduction to abstract algebra. The history of mathematics and science has been immeasurably enriched by his *Science Awakening, Sources of Quantum Mechanics* and his recent *The Birth of Astronomy*.

Daniel Gallin and Edwin Shapiro ("Optimal Investment under Risk") are professors of Mathematics and Business Administration, respectively, at the University of San Francisco. Daniel Gallin holds a Ph.D. in mathematical logic from the University of California at Berkeley (1972) and Edwin Shapiro specialized in analysis at the University of Pittsburgh where he received his Ph.D. in 1962. Their search for some interesting classroom problems on linear programming led them, quite unexpectedly, to the non-convex example treated in this article. Leonard, their protagonist, is now living comfortably in a small Caribbean tax haven (with probability .90).

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ARTICLES

Hamilton's Discovery of Quaternions

Contemporary sources describe Hamilton's trail from repeated failures at multiplying triplets to the intuitive leap into the fourth dimension.

B. L. VAN DER WAERDEN
University of Zurich

Introduction

The ordinary complex numbers $(a + ib)$ (or, as they were formerly written, $a + b\sqrt{-1}$) are added and multiplied according to definite rules. The rule for multiplication reads as follows:

> First multiply according to the rules of high school algebra:
>
> $$(a + ib)(c + id) = ac + adi + bci + bdi^2$$
>
> and then replace i^2 by (-1):
>
> $$(a + ib)(c + id) = (ac - bd) + (ad + bc)i.$$

Complex numbers can also be defined as couples (a, b). The product of two couples (a, b) and (c, d) is defined as the couple $(ac - bd, ad + bc)$. The couple $(1, 0)$ is called 1, the couple $(0, 1)$ is called i. Then we also have the result

$$i^2 = (0,1)(0,1) = (-1,0) = -1.$$

By means of this definition the "imaginary unit" $i = \sqrt{-1}$ loses all of its mystery: i is simply the couple $(0, 1)$.

The **quaternions** $a + bi + cj + dk$ which William Rowan Hamilton discovered on the 16th of October, 1843, are multiplied according to fixed rules, in analogy to the complex numbers; that is to say:

$$i^2 = j^2 = k^2 = -1,$$

$$ij = k, \qquad jk = i, \qquad ki = j,$$

$$ji = -k, \qquad kj = -i, \qquad ik = -j.$$

They can also be defined as quadruples (a, b, c, d). Quaternions form a **division algebra;** that is, they cannot only be added, subtracted, and multiplied, but also divided (excluding division by zero). All rules of calculation of high school school algebra hold; only the commutative law $AB = BA$ does not hold since ij is not the same as ji.

Originally published in German as "Hamilton's Entdeckung der Quaternionen" in *Veröffentlichungen der Joachim Jungius-Gesellschaft Hamburg* by Vandenhoeck & Ruprecht, Gottingen, 1974. Translated into English by F. V. Pohle, Adelphi University, and published here by permission of the original publisher Vandenhoeck & Ruprecht.

How did Hamilton arrive at these multiplication rules? What was his problem and how did he find the solution? We are accurately informed about these matters in documents and papers which appear in the third volume of Hamilton's collected *Mathematical Papers* [**3**]:

> *First*, through an entry in Hamilton's Note Book dated 16 October 1843 [**3**, pp. 103–105];
>
> *Second*, through a letter to John Graves of the 17th of October 1843 [**3**, pp. 106–110];
>
> *Third*, through a paper in the Proceedings of the Royal Irish Academy (**2** (1844) 424–434) presented on the 13th of November 1843 [**3**, pp. 111–116];
>
> *Fourth*, through the detailed Preface to Hamilton's "Lectures on Quaternions", dated June 1853 [**3**, pp. 117–155, in particular pp. 142–144];
>
> *Fifth*, through a letter to his son Archibald which Hamilton wrote shortly before his death, that is shortly before the 2nd of September 1865 [**3**, pp. xv–xvi].

We can follow exactly each of Hamilton's steps of thought through all of these documents. This is a rare occurrence in which we can observe what flashed across the mind of a mathematician as he posed the problem, as he approached the solution step by step and then through a lightning stroke so modified the problem that it became solvable.

A brief history of complex numbers

Expressions of the form $A+\sqrt{-B}$ had already been encountered in the middle ages in the solution of quadratic equations. They were called "impossible solutions" or *numeri surdi*: absurd numbers. The negative numbers too were called "impossible." Cardan used numbers $A+\sqrt{-B}$ in the solution of equations of the third degree in the *casus irreducibilis* in which all three roots are real. Bombelli showed that it was possible to calculate with expressions such as $A+\sqrt{-B}$ without contradiction, but he did not like them: he called them "sophistical" and apparently without value. The expression "imaginary number" stems from Descartes.

Euler had no scruples about operating altogether freely with complex numbers. He proposed formulas such as $\cos\alpha = (1/2)(e^{i\alpha}+e^{-i\alpha})$. The geometric representation of complex numbers as vectors or as points in a plane stems from Argand (1813), Warren (1828) and Gauss (1832).

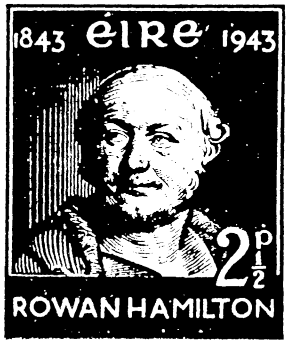
The first named, Argand, defined the complex numbers as directed segments in the plane. He took the basis vectors 1 and i as mutually perpendicular unit vectors. Addition is the usual vector addition, with which Newton made us familiar (the parallelogram law of velocities or of forces). The length of a vector was denoted at that time by the term "modulus", the angle of the vector with the positive x-axis as the "argument" of the complex number. Multiplication of complex numbers, according to Argand, then takes place so that the moduli are multiplied and the arguments are added. Independently of Argand, Warren and Gauss also represented complex numbers geometrically and interpreted their addition and multiplication geometrically.

"Papa, can you multiply triplets?"

Hamilton knew and used the geometric representation of complex numbers. In his published papers, however, he emphasized the definition of complex numbers as the couple (a,b) which followed definite rules for addition and multiplication. Related to that, Hamilton posed this problem to himself: *To find how number-triplets* (a,b,c) *are to be multiplied in analogy to couples* (a,b).

For a long time Hamilton had hoped to discover the multiplication rule for triplets, as he himself stated. But in October 1843 this hope became much stronger and more serious. He put it this way in a letter to his son Archibald [**3**, p. xv]:

SIR WILLIAM ROWAN HAMILTON, a child prodigy whose maturity was all that his childhood promised, was born in Dublin, Ireland, in 1805. He was literate in seven languages and knowledgeable in half a dozen more. In 1827 while still an undergraduate Hamilton was appointed Andrews Professor of Astronomy and Superintendent of the Observatory, and soon afterwards Astronomer Royal, a position he held for the rest of his extraordinarily productive life. His work in dynamics is probably most well-known today. "The Hamiltonian principle has become the cornerstone of modern physics", said Erwin Schrödinger, "the thing with which a physicist expects *every* physical phenomenon to be in conformity." Hamilton's other major discovery is the system of quaternions. The flash of insight which produced this discovery occurred in 1843 and is described in the accompanying article. A century later the Irish government commemorated this achievement with the stamp pictured at the right.



> "... the desire to discover the law of multiplication of triplets regained with me a certain strength and earnestness, ..."

In analogy to the complex numbers $(a+ib)$ Hamilton wrote his triplets as $(a+bi+cj)$. He represented his unit vectors $1, i, j$ as mutually perpendicular "directed segments" of unit length in space. Later Hamilton himself used the word vector, which I also shall use in the following. Hamilton then sought to represent products such as $(a+bi+cj)(x+yi+zj)$ again as vectors in the same space. He required, first, that it be possible to multiply out term by term; and second, that the length of the product of the vectors be equal to the product of the lengths. This latter rule was called the "law of the moduli" by Hamilton.

Today we know that the two requirements of Hamilton can be fulfilled only in spaces of dimensions 1, 2, 4 and 8. This was proved by Hurwitz [**5**]. Therefore Hamilton's attempt in three dimensions had to fail. His profound idea was to continue to 4 dimensions since all of his attempts in 3 dimensions failed to reach the goal.

In the previously mentioned letter to his son, Hamilton wrote about his first attempt:

> "Every morning in the early part of the above-cited month [October 1843], on my coming down to breakfast, your brother William Edwin and yourself used to ask me, 'Well, Papa, can you multiply triplets?' Whereto I was always obliged to reply, with a sad shake of the head, 'No, I can only add and subtract them.'"

From the other documents we learn more precisely about Hamilton's first attempts. To fulfill the "law of the moduli" at least for the complex numbers $(a+ib)$, Hamilton set $ii=-1$, as for ordinary complex numbers, and similarly so that the law would also hold for the numbers $(a+cj)$, $jj=-1$. But what was ij and what was ji? At first Hamilton assumed $ij=ji$ and calculated as follows:

$$(a+ib+jc)(x+iy+jz)=(ax-by-cz)+i(ay+bx)+j(az+cx)+ij(bz+cy).$$

Now, he asked, what is one to do with ij? Will it have the form $\alpha+\beta i+\gamma j$?

First attempt. The square of ij had to be 1, since $i^2=-1$ and $j^2=-1$. Therefore, wrote Hamilton, in this attempt one would have to choose $ij=1$ or $ij=-1$. But in neither of these two cases will the law of the moduli be fulfilled, as calculation shows.

Second attempt. Hamilton considered the simplest case

$$(a+ib+jc)^2=a^2-b^2-c^2+2iab+2jac+2ijbc.$$

Then he calculated the sum of the squares of the coefficients of 1, i, and j on the right hand side and found

$$(a^2-b^2-c^2)^2+(2ab)^2+(2ac)^2=(a^2+b^2+c^2)^2.$$

Therefore, he said, the product rule is fulfilled if we set $ij = 0$. And further: if we pass a plane through the points 0, 1, and $a + ib + jc$, then the construction of the product according to Argand and Warren will hold in this plane: the vector $(a + bi + cj)^2$ lies in the same plane and the angle which this vector makes with the vector 1 is twice as large as the angle between the vectors $(a + bi + cj)$ and 1. Hamilton verified this by computing the tangents of the two angles.

Third attempt. Hamilton reports that the assumption $ij = 0$, which he made in the second attempt, subsequently did not appear to be quite right to him. He writes in the letter to Graves [**3**, p. 107]:

> "Behold me therefore tempted for a moment to fancy that $ij = 0$. But this seemed odd and uncomfortable, and I perceived that the same suppression of the term which was *de trop* might be attained by assuming what seemed to me less harsh, namely that $ji = -ij$. I made therefore $ij = k$, $ji = -k$, reserving to myself to inquire whether k was 0 or not."

Hamilton was entirely right in giving up the assumption $ij = 0$ and taking instead $ij = -ji$. For example, if $ij = 0$ then the modulus of the product ij would be zero, which would contradict the law of the moduli.

Fourth attempt. Somewhat more generally, Hamilton multiplied $(a + ib + jc)$ and $(x + ib + jc)$. In this case the two segments which are to be multiplied also lie in one plane, that is, in the plane spanned by the points 0, 1, and $ib + jc$. The result of the multiplication was $ax - b^2 - c^2 + i(a + x)b + j(a + x)c + k(bc - bc)$. Hamilton concluded from this calculation [**3**, p. 107] that:

> "... the coefficient of k still vanishes; and $ax - b^2 - c^2$, $(a + x)b$, $(a + x)c$ are easily found to be the correct coordinates of the *product-point* in the sense that the rotation from the unit line to the radius vector of a, b, c being added in its own plane to the rotation from the same unit-line to the radius vector of the other factor-point x, b, c conducts to the radius vector of the lately mentioned product-point; and that this latter radius vector is in length the product of the two former. Confirmation of $ij = -ji$; but no information yet of the value of k."

The leap into the fourth dimension

After this encouraging result Hamilton ventured to attack the general case. ("Try boldly then the general product of two triplets, ..." [**3**, p. 107].) He calculated

$$(a + ib + jc)(x + iy + jz) = (ax - by - cz) + i(ay + bx) + j(az + cx) + k(bz - cy).$$

In an exploratory attempt he set $k = 0$ and asked: Is the law of the moduli satisfied? In other words, does the identity

$$(a^2 + b^2 + c^2)(x^2 + y^2 + z^2) = (ax - by - cz)^2 + (ay + bx)^2 + (az + cx)^2$$

hold?

> "No, the first member exceeds the second by $(bz - cy)^2$. But this is just the square of the coefficient of k, in the development of the product $(a + ib + ic)(x + iy + jz)$, if we grant that $ij = k$, $ji = -k$, as before."

And now comes the insight which gave the entire problem a new direction. In the letter to Graves [**3**, p. 108], Hamilton emphasized the insight:

> "And here there dawned on me the notion that we must admit, in some sense, a *fourth dimension* of space for the purpose of calculating with triplets;"

This fourth dimension appeared as a "paradox" to Hamilton himself and he hastened to transfer the paradox to algebra [**3**, p. 108]:

> "...; or transferring the paradox to algebra, [we] must admit a *third* distinct imaginary symbol k, not to be confounded with either i or j, but equal to the product of the first as multiplier, and the second as multiplicand; and therefore [I] was led to introduce *quaternions* such as $a + ib + jc + kd$, or (a, b, c, d)."

Hamilton was not the first to think about a multi-dimensional geometry. In a footnote to the letter to Graves he wrote:

> "The writer has this moment been informed (in a letter from a friend) that in the Cambridge Mathematical Journal for May last [1843] a paper on Analytical Geometry of n dimensions has been published by Mr. Cayley, but regrets he does not yet know how far Mr. Cayley's views and his own may resemble or differ from each other."

"This moment" can in this connection only mean the same day in which he wrote the letter to Graves. In the Note Book of the 16th of October 1843 there is no mention of the paper by Cayley. Hamilton therefore appears to have arrived at the concept of a 4-dimensional space independently of Cayley.

After Hamilton had introduced $ij = -ji = k$ as a fourth independent basis vector, he continued the calculation [**3**, p. 108]:

> "I saw that we had probably $ik = -j$, because $ik = iij$, and $i^2 = -1$; and that in like manner we might expect to find $kj = ijj = -i$;"

From the use of the word "probably" it can be seen how cautiously Hamilton continued. He scarcely trusted himself to apply the associative law $i(ij) = (ii)j$ because he was not yet certain if the associative law held for quaternions. Likewise Hamilton could have used the associative law to determine ki:

$$ki = -(ji)i = -j(ii) = (-j)(-i) = j.$$

Instead he applied a conclusion by analogy. He wrote [**3**, p. 108]

> "...; from which I thought it likely that $ki = j$, $jk = i$, because it seemed likely that if $ji = -ij$, we should have also $kj = -jk$, $ik = -ki$."

Finally k^2 had to be determined. Hamilton again proceeded cautiously:

> "And since the order of multiplication of these imaginaries is not indifferent, we cannot infer that k^2, or $ijij$, is $= +1$, because $i^2 \times j^2 = (-1)(-1) = +1$. It is more likely that $k^2 = ijij = -iijj = -1$."

This last assumption $k^2 = -1$, asserts Hamilton, is also necessary if we wish to fulfill the "law of the moduli." He carried this out and concluded [**3**, p. 108]:

> "My assumptions were now completed, namely,
>
> $$i^2 = j^2 = k^2 = -1$$
> $$ij = -ji = k$$
> $$jk = -kj = i$$
> $$ki = -ik = j."$$

And now Hamilton tested if the law of the moduli was actually satisfied.

> "But I considered it essential to try whether these equations were consistent with the law of moduli, ..., without which consistence being verified, I should have regarded the whole speculation as a failure."

He therefore multiplied two arbitrary quaternions according to the rules just formulated

$$(a, b, c, d)(a', b', c', d') = (a'', b'', c'', d''),$$

calculated (a'', b'', c'', d'') and formed the sum of the squares

$$(a'')^2 + (b'')^2 + (c'')^2 + (d'')^2$$

and found to his great joy that this sum of squares actually was equal to the product

$$(a^2 + b^2 + c^2 + d^2)(a'^2 + b'^2 + c'^2 + d'^2).$$

In Hamilton's letter to his son we learn even more about the external circumstances which befell him at this flash of insight. Immediately after the previously cited words, "No, I can only add and subtract them." Hamilton continued [3, p. xx–xvi]:

> "But on the 16th day of the same month [October 1843]—which happened to be a Monday and a Council day of the Royal Irish Academy—I was walking in to attend and preside, and your mother was walking with me, along the Royal Canal, to which she had perhaps been driven; and although she talked with me now and then, yet an under-current of thought was going on in my mind, which gave at last a result, whereof it is not too much to say that I felt at once the importance. An electric circuit seemed to close; and a spark flashed forth, the herald (as I foresaw immediately) of many long years to come of definitely directed thought and work, by myself if spared, and at all events on the part of others, if I should ever be allowed to live long enough distinctly to communicate the discovery. I pulled out on the spot a pocket-book, which still exists, and made an entry there and then. Nor could I resist the impulse—unphilosophical as it may have been—to cut with a knife on a stone of Brougham Bridge, as we passed it, the fundamental formula with the symbols i, j, k;
>
> $$i^2 = j^2 = k^2 = ijk = -1,$$
>
> which contains the solution of the Problem, but of course as an inscription, has long since mouldered away."

The entry in the pocket book is reproduced on the title page of [3]: it contains the formulas

$$i^2 = j^2 = k^2 = -1$$
$$ij = k, \qquad jk = i, \qquad ki = j$$
$$ji = -k, \qquad kj = -i, \qquad ik = -j.$$

I assume as likely that before his walk Hamilton had already written on a piece of paper the result of the somewhat tiresome calculation which showed that the sum of squares

$$(ax - by - cz)^2 + (ay + bx)^2 + (az + cx)^2$$

still lacked $(bz - cy)^2$ compared with the product

$$(a^2 + b^2 + c^2)(x^2 + y^2 + z^2).$$

What then happened immediately before and during that remarkable walk along the Royal Canal, he described again on the same day in his Note Book, as follows:

> "I believe that I now remember the order of my thought. The equation $ij = 0$ was recommended by the circumstances that
>
> $$(ax - y^2 - z^2)^2 + (a + x)^2(y^2 + z^2) = (a^2 + y^2 + z^2)(x^2 + y^2 + z^2).$$
>
> I therefore tried whether it might not turn out to be true that
>
> $$(a^2 + b^2 + c^2)(x^2 + y^2 + z^2) = (ax - by - cz)^2 + (ay + bx)^2 + (az + cx)^2,$$
>
> but found that this equation required, in order to make it true, the addition of $(bz - cy)^2$ to the second member. This *forced* on me the non-neglect of ij, and *suggested* that it might be equal to k, a new imaginary."

By underscoring the italicized words *forced* and *suggested* Hamilton emphasized that he was concerned with two entirely different facts. The first was a compelling logical conclusion, which came immediately out of the calculation: it was not possible to set ij equal to zero, since then the law of the moduli would not hold. The second fact was an insight which came over him in a flash at the canal ("an electric circuit seemed to close, and a spark flashed forth"), that is, that ij could be taken to be a new imaginary unit.

After the insight was once there, everything else was very simple. The calculations $ik = iij = -j$ and $kj = ijj = -i$ could be made easily enough by Hamilton in his head. The assumptions $ki = -ik = j$ and $jk = -kj = i$ were immediate. And k^2 could be easily calculated too: $k^2 = ijij = -iijj = -1$.

And so during his walk Hamilton also discovered the rules of calculation which he entered into the pocket book. The pocket book also contains the formulas for the coefficients of the product

$$(a + bi + cj + dk)(\alpha + \beta i + \gamma j + \delta k),$$

that is,

$$a\alpha - b\beta - c\gamma - d\delta$$

$$a\beta + b\alpha + c\delta - d\gamma$$

$$a\gamma - b\delta + c\alpha + d\beta$$

$$a\delta + b\gamma - c\beta + d\alpha$$

as well as the sketch for the verification of the fact that in the sum of the squares of these coefficients all mixed terms (such as $ad\alpha\delta$) cancel and only $(a^2 + b^2 + c^2 + d^2)(\alpha^2 + \beta^2 + \gamma^2 + \delta^2)$ remains. In the Note Book of the same day everything was again completely restated.

Octonions

The letter to Graves in which Hamilton announced the discovery of quaternions was written on the 17th of October 1843, one day after the discovery. The seeds, which Hamilton sowed, fell upon fertile soil, since in December 1843 the recipient John T. Graves already found a linear algebra with 8 unit elements 1, i, j, k, l, m, n, o, the algebra of *octaves* or *octonions*. Graves defined their multiplication as follows [**3**, p. 648]:

$$i^2 = j^2 = k^2 = l^2 = m^2 = n^2 = o^2 = -1$$

$$i = jk = lm = on = -kj = -ml = -no$$

$$j = ki = ln = mo = -ik = -nl = -om$$

$$k = ij = lo = nm = -ji = -ol = -mn$$

$$l = mi = nj = ok = -im = -jn = -ko$$

$$m = il = oj = kn = -li = -jo = -nk$$

$$n = jl = io = mk = -lj = -oi = -km$$

$$o = ni = jm = kl = -in = -mj = -lk.$$

In this system the "law of the moduli" also holds:

$$(1) \qquad (a_1^2 + \cdots + a_8^2)(b_1^2 + \cdots + b_8^2) = (c_1^2 + \cdots + c_8^2)$$

Hamilton answered on the 8th of July 1844 [**3**, p. 650]. He noted to Graves that the associative law $A \cdot BC = AB \cdot C$ clearly held for quaternions but not for octaves.

Octaves were rediscovered by Cayley in 1845; because of this they are also known as *Cayley numbers*. Graves also made an attempt with 16 unit elements but it was unsuccessful. It could not succeed since we know today that identities of the form (1) are only possible for sums of 1, 2, 4 and 8 squares. I should like to close with a brief comment about the history of these identities.

Product formulas for the sums of squares

It is likely that the "law of the moduli" for complex numbers was already known to Euler:

$$(a^2 + b^2)(c^2 + d^2) = (ac - bd)^2 + (ad + bc)^2.$$

A similar formula for the sum of 4 squares

$$(a_1^2 + \cdots + a_4^2)(b_1^2 + \cdots + b_4^2) = (c_1^2 + \cdots + c_4^2)$$

was discovered by Euler; the formula is stated in a letter from Euler to Goldbach on May 4th, 1748 [**4**]. The formula (1) for 8 squares, which Graves and Cayley proved by means of octonions, was previously found by Degen (1818) [**6**]. Degen erroneously thought that he could generalize the theorem to 2^n squares.

The problem, which started with Hamilton, reads: can two triplets (a, b, c) and (x, y, z) be so multiplied that the law of the moduli holds? In other words: is it possible so to define (u, v, w) as bilinear functions of (a, b, c) and (x, y, z) that the identity

$$(a^2 + b^2 + c^2)(x^2 + y^2 + z^2) = (u^2 + v^2 + w^2) \tag{2}$$

results?

The first to show the impossibility for this identity was Legendre. In his great work *Théorie des nombres* he remarked on page 198 that the numbers 3 and 21 can easily be represented rationally as sums of three squares:

$$3 = 1 + 1 + 1,$$

$$21 = 16 + 4 + 1,$$

but the product $3 \times 21 = 63$ cannot be so represented, since 63 is an integer of the form $(8n + 7)$. It follows from this that an identity of the form (2) is impossible, to the extent that it is assumed that (u, v, w) are bilinear forms in (a, b, c) and (x, y, z) with rational coefficients. If Hamilton had known of this remark by Legendre he would probably have quickly given up the search to multiply triplets. Fortunately he did not read Legendre: he was self-taught.

The question for which values of n a formula of the kind

$$(a_1^2 + \cdots + a_n^2)(b_1^2 + \cdots + b_n^2) = (c_1^2 + \cdots + c_n^2)$$

is possible, was finally decided by Hurwitz in 1898. With the help of matrix multiplication he proved (in [**5**]) that $n = 1, 2, 4$ and 8 are the only possibilities. For further historical accounts the reader may refer to [**1**] or [**2**].

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The sun never sets on mathematics

> The discoveries of Newton have done more for England and for the race, than has been done by whole dynasties of British monarchs; and we doubt not that in the great mathematical birth of 1843, the Quaternions of Hamilton, there is as much real promise of benefit to mankind as in any event of Victoria's reign.
>
> — Thomas Hill

Optimal Investment Under Risk

The useful interplay of linear programming and probability theory forms the basis for an optimal strategy for a cautious investor.

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It has become increasingly common in recent years to find linear programming and finite probability theory included in the undergraduate mathematics curriculum as examples of mathematical techniques that have widespread application in business and economics. The purpose of this note is to illustrate the interplay of these two subjects by means of a specific investment problem: to determine the best allocation of a fixed amount of capital among several investment opportunities, striking a suitable balance between risk and expected gain. The solution we present uses only elementary ideas, such as those ordinarily included in a standard course in finite mathematics.

Consider a potential investor, whom we shall call Leonard. Patient and frugal, Leonard has finally managed to accumulate a total of \$12,000. Lately, though, he has grown alarmed at the soaring rate of inflation and the threat it poses to his life savings. He is only too well aware that he is in effect allowing his principal to slowly dissipate by leaving it in Rebozo Savings and Loan at 5% interest. Consequently Leonard has decided that he is ready to take a chance on an investment with a potentially high return, provided that the risk involved is tolerable. He is currently interested in three possibilities: a Broadway musical production (A), molybdenum futures (B), and an oil development scheme (C). Ever cautious, Leonard has investigated the past performance of ventures similar to A, B and C, and his research has produced the following information: Broadway musicals have a 25% failure rate, but when they are successful one can expect, on the average, to double one's money. To simplify his analysis, Leonard makes the assumption that there are only two possible outcomes for investment A: Failure (F), in which case he loses his investment, and success (S), in which case he realizes a 100% return. The probabilities of F and S are .25 and .75, respectively. For molybdenum futures, Leonard estimates the failure rate at 20%, with an average return of 75% when successful, while oil development schemes have a 10% failure rate and a potential return of 50%.

Leonard recognizes that the three investment opportunities present him with a choice between lower risk and potentially higher return, but he is at a loss for a single, uniform standard by which he can compare the three options. He consults his friend Max, a mathematician, who suggests that he compare the *expected* return for each investment. If he puts x dollars into investment A, he will realize a net gain of x dollars with probability .75, and a net gain of $-x$ dollars (i.e., the loss of his investment) with probability .25. His expected net gain is therefore $.75x + .25(-x) = .50x$ dollars, so that investment A offers an expected return of 50%. Applying a similar analysis to investments B and C, Leonard obtains:

Investment	Risk	Return	Expected Return
A	.25	100%	50%
B	.20	75%	40%
C	.10	50%	35%

Max advises Leonard to put his savings into investment A, since it has the highest expected return. Leonard remains reticent, however. Expected return, he argues, may be a useful concept for professional speculators who intend to make such investments day after day — one can then reasonably expect the law of averages to apply. But Leonard's case is different. If he follows Max's advice he runs a 25% risk of losing his entire life savings, an understandably dreadful prospect. Max then poses an intriguing question: Just how big a risk is his friend willing to take? After a moment's reflection, Leonard replies that he will accept no less than 90% certainty of at least breaking even on his investment. Fortunately, Max happens to recall his broker's dictum that diversification reduces risk, and he suggests that they try to find a judicious way to divide Leonard's capital among the three investments.

Suppose he invests x, y and z dollars, respectively, in A, B and C. His expected net gain, in dollars, will then be

$$E(G) = .50x + .40y + .35z,$$

which he would of course like to maximize, subject to the constraints

(1) $$x, y, z \geqq 0$$

(2) $$x + y + z \leqq 12{,}000$$

and also the risk constraint

(3) $$P(G \geqq 0) \geqq .90,$$

which expresses Leonard's condition that the probability of realizing a non-negative net gain must be at least .90. How can constraint (3) be expressed in terms of x, y and z? Suppose we make the additional assumption that the success or failure of any one of the investments is independent of the success or failure of the others (they are, after all, quite unrelated ventures). We then have eight possible outcomes for the three investments, and we can calculate their respective probabilities:

	A	B	C	Probability	Actual Net Gain G
1.	S	S	S	.540	$1.00x + .75y + .50z$
2.	S	S	F	.060	$1.00x + .75y - z$
3.	S	F	S	.135	$1.00x - y + .50z$
4.	S	F	F	.015	$1.00x - y - z$
5.	F	S	S	.180	$-x + .75y + .50z$
6.	F	S	F	.020	$-x + .75y - z$
7.	F	F	S	.045	$-x - y + .50z$
8.	F	F	F	.005	$-x - y - z$

For example, the probability that A will fail but B and C will succeed is $P(FSS) = (.25)(.80)(.90) = .180$, and in this case Leonard stands to lose x dollars but gain $.75y + .50z$ dollars, for a total net gain $G = -x + .75y + .50z$. Since the eight cases are mutually exclusive, we have

$$P(G \geqq 0) = P(SSS) \cdot P(G \geqq 0 | SSS) + \cdots + P(FFF) \cdot P(G \geqq 0 | FFF),$$

where, e.g., $P(G \leqq 0 | FSS)$ is the conditional probability that $G \geqq 0$ will occur, given that FSS occurs. Since $G = -x + .75y + .50z$ in this case, and this quantity depends only on x, y and z, the conditional probability in question will be either 0 or 1. But the same is true of the remaining seven cases, and we can therefore write

$$(*) \qquad P(G \geqq 0) = p_1\varepsilon_1 + p_2\varepsilon_2 + \cdots + p_8\varepsilon_8,$$

where, e.g., $p_5 = P(FSS) = .180$, and ε_5 takes the value 1 or 0 according as $-x + .75y + .50z \geqq 0$ or < 0. Equivalently,

$$(**) \qquad P(G \geqq 0) = \Sigma_{i \in I}\, p_i,$$

where $i \in I$ iff $\varepsilon_i = 1$. For the investment plan or "portfolio" defined by $x = y = 5000$, $z = 2000$, for instance, we find that $I = \{1, 2, 3\}$ and hence $P(G \geqq 0) = p_1 + p_2 + p_3 = .735$.

Now, let us assume that a given portfolio $P = (x, y, z)$ satisfies the risk constraint $P(G \geqq 0) \geqq .90$. Then (**) shows that we must have $i \in I$ for $i = 1, 3$ and 5, since $p_i > .10$ for each of these values. Moreover, I must contain either 2 or 7, since otherwise $I \subseteq \{1, 3, 4, 5, 6, 8\}$ and hence

$$P(G \geqq 0) \leqq p_1 + p_3 + p_4 + p_5 + p_6 + p_8 = .895 < .90.$$

We see from this that one of two conditions must obtain: Either $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_5 = 1$ or else $\varepsilon_1 = \varepsilon_3 = \varepsilon_5 = \varepsilon_7 = 1$. Conversely, if either of these conditions holds then by (*), $P(G \geqq 0) \geqq .90$. The risk constraint (3) is therefore equivalent to the *disjunction* of two different sets of joint linear inequalities, viz., the set

(3.1a) $\qquad 1.00x + .75y + .50z \geqq 0$

(3.1b) $\qquad 1.00x + .75y - z \geqq 0$

(3.1c) $\qquad 1.00x - y + .50z \geqq 0$

(3.1d) $\qquad -x + .75y + .50z \geqq 0,$

which we denote by (3.1), and the set

(3.2a) $\qquad 1.00x + .75y + .50z \geqq 0$

(3.2b) $\qquad 1.00x - y + .50z \geqq 0$

(3.2c) $\qquad -x + .75y + .50z \geqq 0$

(3.2d) $\qquad -x - y + .50z \geqq 0,$

which we denote by (3.2). We can therefore formulate Leonard's investment problem as follows: *Maximize* $E(G) = .50x + .40y + .35z$ *subject to the condition that either the constraints* (1), (2) *and* (3.1) *hold, or else* (1), (2) *and* (3.2) *hold.*

Viewed geometrically, the problem here is to maximize a linear objective function over a constraint set S which is a union $S_1 \cup S_2$ of convex polyhedral sets. There is no difficulty in solving a problem of this type, however: We have only to maximize the function over S_1 and over S_2 separately — obtaining maximum values at points $P_1 \in S_1$ and $P_2 \in S_2$ — and then choose whichever point gives the greater value of the objective function. If we apply the simplex algorithm with the constraints (1), (2) and (3.1), we obtain a maximum value $E(G) = 88{,}800/17 = 5223.53$ at the point $P_1 = (x_1, y_1, z_1)$, where $x_1 = 84{,}000/17 = 4941.18$, $y_1 = 96{,}000/17 = 5647.06$, and $z_1 = 24{,}000/17 = 1411.76$. For the constraints (1), (2) and (3.2), the simplex algorithm gives a maximum value $E(G) = 4800$ at the point $P_2 = (x_2, y_2, z_2)$, where $x_2 = 4000$, $y_2 = 0$, and $z_2 = 8000$. Since $E(G)$ takes on the greater value at P_1, this point is the desired solution.

Leonard has good reason to accept P_1 as an optimal investment plan. It offers an expected return of approximately 43.53%, which is more than the expected return on the intermediate investment B. At the same time, we know that P_1 satisfies the risk constraint (3), so it is no less safe than the safest investment C. In fact (as Max's broker suggested) it turns out to be even safer: Since P_1 makes $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_5 = 1$, with $\varepsilon_i = 0$ otherwise, we find from (*) that the probability of breaking even is actually .915. By allocating his capital among the three investments in this optimal way, therefore, Leonard can enjoy a reassuring 91.5% certainty of preserving his life savings.

The investment model we have illustrated here can of course be generalized. We can allow any finite number of investment opportunities A_i $(i = 1, 2, \ldots, n)$, each having a given return ρ_i, expressed as a fraction of the amount invested. In general, ρ_i will be a random variable taking on known values $(\geqq -1)$ with known probabilities. For our earlier investment B, for example, the return ρ takes on the values -1 and .75 with respective probabilities .20 and .80. If the investments cannot be assumed to be independent, we must also obtain the joint distribution function for $\rho_1, \rho_2, \ldots, \rho_n$. The risk constraint $P(G \geqq 0) \geqq .90$ can be replaced by any condition or set of conditions of the form $P(G \geqq a_0) \geqq q_0$; for example, an investor might be willing to accept 95% certainty that he would lose no more than \$2000 $(a_0 = -2000,\ q_0 = .95)$.

If we let $e_i = E(\rho_i)$, x_i = the dollar amount invested in A_i, and b = the total amount of capital, and we introduce the row vectors $\rho = (\rho_1, \ldots, \rho_n)$, $e = (e_1, \ldots, e_n)$ and the column vector $x = (x_1, \ldots, x_n)^T$, then $\rho \cdot x$ represents the actual net gain G, $e \cdot x$ represents the expected net gain $E(G)$, and the general problem becomes: *Maximize* $E(G) = e \cdot x$ *subject to the constraints*

(i) $x_i \geqq 0$ $(i = 1, \ldots, n)$

(ii) $\sum x_i \leqq b$

(iii) $P(\rho \cdot x \geqq a_j) \geqq q_j$ $(j = 1, \ldots, m)$.

The practical solution of such a stochastic, or chance-constrained, program (CCP) will in general require techniques more sophisticated than those employed in our simplified example. The interested reader is referred to Wagner [7] and Vajda [6] for a general introduction to stochastic programming, and to [1], [3] and [5] for more detailed technical accounts. Some of the practical questions connected with the implementation of portfolio selection models are discussed in Markowitz [4] and Fried [2].

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Win some, lose some

If you had it to do over, would you change anything? "Yes, I wish I had played the black instead of the red at Cannes and Monte Carlo."

— WINSTON CHURCHILL

Twelve and its Totitives

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Results concerning the prime numbers are amongst the most surprising in the whole of mathematics. One such example is the subject of an essay in a delightful collection by Otto Toeplitz and Hans Rademacher [1] under the title *The Enjoyment of Mathematics.*

Following Sylvester, we call those positive integers not greater than and relatively prime to some given integer, its totitives. Thus, for example, 1, 5, 7 and 11 are the totitives of 12, whilst 1, 7, 11, 13, 17, 19, 23 and 29 are those of 30. The totitives of a number play an important role in much number theory and are still a favorite topic of research. The theorem, of which Toeplitz and Rademacher give an elementary proof, is that 30 is the largest integer all of whose totives are primes, a result first proved in 1893 by Schantunowsky [2]. A moment's thought shows that 30 is also the largest integer from which, no matter which of its totitives is subtracted, the result is always a prime. The obvious problem arises of finding the largest integer, should it exist, to which each of its totitives can be added to obtain a prime. We conjecture that *the largest integer to which each of its totitives can be added to obtain a prime is* 12. Though we cannot prove this conjecture, we can show that a largest such integer does indeed exist. In this paper we shall show that the conjecture can be deduced from a likely but as yet unproved inequality of Landau.

Let $\pi(n)$ denote the number of primes (including 1) not greater than n. Landau's inequality is: $\pi(2n)-2\pi(n)<0$. What we shall prove is that if this holds for all positive integers, then 12 is the largest integer with the required property. Landau [3] proved the inequality for sufficiently large integers, and Segal [4] has shown it true for all integers up to 50,540, so one is inclined to believe it to be true in general. However, if it is not, a slight modification of the argument shows that such a largest integer exists, although in this case it need not be 12.

Throughout let $Q_k = 1\cdot 2\cdot 3\cdots P_{k-1}\cdot P_k$ denote the product of the first k primes and let S_k be the set of positive integers s such that $Q_k \leqq s < Q_{k+1}$. We shall make use of Bertrand's Postulate that between any number and its double there always lies a prime. An elementary proof of this is given in [5]. We begin with the following result:

LEMMA. *For all $k \geqq 5$, every element of S_k has at least k composite totitives.*

Proof. Assume that the k composite numbers below, which we choose for reasons which will soon become apparent, are smaller than Q_k for $k \geqq 5$:

$$(*)\qquad P_{k+1}P_{k+2},\ P_{k+1}P_{k+3},\ P_{k+1}P_{k+4},\ P^2_{k+1},\ P^2_{k+2},\ P^2_{k+3},\ \ldots,\ P^2_{2k-3}.$$

From these it is possible to construct k composite totitives of any $s \in S_k$ as follows: If $s = Q_k$, the numbers $(*)$ themselves will suffice. If $s > Q_k$, notice that s has at most k distinct prime factors so that if any prime in the set $\{P_{k+1}, P_{k+2}, \ldots, P_{2k-3}\}$ of all prime factors of the numbers $(*)$ is also a factor of s, we can replace it by one from the set $\{1, 2, \ldots, P_k\}$ which is not a factor of s. If we make the corresponding replacements in the list $(*)$, then the resulting k numbers are composite totitives of s.

It remains to show that for any $k \geqq 5$, the numbers $(*)$ are all smaller than Q_k. We do this by induction. It is certainly true for $k = 5$. Suppose that it is true for some k. Now consider the list:

$$P_{k+2}P_{k+3},\ P_{k+2}P_{k+4},\ P_{k+2}P_{k+5},\ P^2_{k+2},\ P^2_{k+3},\ \ldots,\ P^2_{2k-1}.$$

By Bertrand's Postulate, no number in the new list is greater than 8 times the largest of the numbers (*), whilst $Q_{k+1} > 8Q_k$ because $P_{k+1} > 8$. So the numbers in the new list are all smaller than Q_{k+1} and the proof of the lemma is complete.

We can now readily show that Landau's inequality implies the conjecture. If $s \in S_k$, then of the $\pi(s)$ primes smaller than s, at most k divide s, so at least $\pi(s) - k$ are relatively prime to s. These, together with the k composite totitives guaranteed by the lemma for $k \geqq 5$, give at least $\pi(s)$ totitives of s. Suppose now that s is a number to which the addition of each totitive t gives a prime $s + t$. If $k \geqq 5$, then there are at least $\pi(s)$ of these primes $s + t$ lying between s and $2s$. So $\pi(2s) \geqq 2\pi(s)$ and this contradicts Landau's inequality. The totitives of 12 are 1, 5, 7, 11 and 12 + 1, 12 + 5, 12 + 7, 12 + 11 are all prime. A check shows that no number between 12 and $Q_5 = 210$ satisfies the required property, so if Landau's inequality holds for all n, then the conjecture is true. Since Landau's inequality holds for sufficiently large n, say $n \geqq N$, then if s is an integer to which the addition of any of its totitives gives a prime, we must have $s < N$.

It may be difficult to decide whether there is a largest integer with the property that each of its prime (or, alternatively, composite) totitives added to it always results in a prime. In both cases I conjecture that there is such an integer. I do not know either whether there are arbitrarily large numbers to which the adding of any of its prime (composite) totitives always results in a composite number. Of course, because of Bertrand's Postulate no integer exists to which the adding of any of its totitives always results in a composite number.

I wish to thank my tutor, Dr. David Tall, for his encouragement and many suggestions which made possible this note.

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Bounds on the Roots of a Polynomial

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It is the purpose of this note to give bounds for the moduli of the roots of a polynomial. From the practical point of view these bounds give an easily calculated annulus in the complex plane within which each root must lie and thus furnish an inclusion theorem [**1**]. Such bounds are useful in developing efficient computer methods for actually finding (approximations to) the roots of polynomials. The proof itself is of some interest as a fairly concrete application of two well-known theorems of numerical linear algebra.

In what follows we use A to denote a matrix with complex entries, A^* to denote the conjugate transpose of A and det A to denote the determinant of A. The conjugate transpose notation applies as well to vectors which are one-rowed or one-columned matrices.

THEOREM. *Let $p(\lambda)=\lambda^n+p_1\lambda^{n-1}+\cdots+p_n$ be a monic polynomial with complex coefficients. Define $\alpha=|p_n|^2$, $\beta=\Sigma_1^n|p_i|^2$, and let μ_1^2 and μ_2^2 be the maximum and mimimum roots (necessarily positive) of $t^2-(1+\beta)t+\alpha=0$. Then if λ is any root of p, we have $\mu_2 \leqq |\lambda| \leqq \mu_1$.*

Proof. Let A be the matrix

$$A=\begin{pmatrix} c_1 & c_2 & \cdots\cdots & & c_n \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & 1 & 0 \end{pmatrix}$$

where $c_i=-p_i$, $1\leqq i\leqq n$. A is known as the companion matrix of p. By a well-known theorem [**2**], $p(\lambda)=\det(\lambda I-A)$; thus the characteristic roots of A coincide with the roots of p. A classical result in matrix theory [**3**, p. 144] states the following: If $\alpha_1\geqq\alpha_2\geqq\cdots\geqq\alpha_n\geqq 0$ are the positive square roots of the characteristic roots of AA^* (the α_i are often called the singular values of A) then any characteristic root λ of A must satisfy $\alpha_n\leqq|\lambda|\leqq\alpha_1$. What we are going to show is that if A is a companion matrix, α_1 and α_n are easily determined.

Let us partition A as

$$A=\begin{pmatrix} v^* & a \\ I & 0\end{pmatrix}$$

where $v^*=[c_1,c_2,\ldots,c_{n-1}]$ and $a=c_n$. Then

$$AA^*=\begin{pmatrix} \beta & v^* \\ v & I\end{pmatrix}$$

where we have used $v^*v+|a|^2=\beta$. A standard determinantal identity [**4**, p. 46] shows that

$$\det(sI-AA^*)=\det(sI-I)[s-\beta-v^*(sI-I)^{-1}v]. \tag{1}$$

Direct computation then shows that

$$\det(sI-AA^*)=(s-1)^{n-2}[(s-\beta)(s-1)-v^*v].$$

Thus we see that $s=1$ is a characteristic root of AA^* of multiplicity at least $n-2$. The remaining roots satisfy the equation $s^2-(1+\beta)s+\beta-v^*v=0$. It is a fact that the two roots of this quadratic equation are the maximum and minimum characteristic roots of AA^*. This follows easily from the classical separation theorem due to Cauchy and improved by Browne [**1**, p. 76] [**3**, p. 119] [**5**, p. 117]: given any nth order Hermitian matrix the characteristic roots of any principal submatrix of order $n-1$ separate the roots of the matrix. Thus if $s_1\geqq s_2\geqq\cdots\geqq s_n$ are the ordered roots of AA^*, then $s_1\geqq 1\geqq s_2\geqq\cdots\geqq 1\geqq s_n\geqq 0$. Since $\alpha_i=\sqrt{s_i}$, and $\alpha=|a|^2=\beta-v^*v$, α_1^2 and α_n^2 must satisfy the equation $t^2-(1=\beta)t+\alpha=0$. This completes the proof of the theorem.

We conclude with some observations about our proof. The determinantal identity used to obtain (1) applies to any suitably partitioned matrix. For the case at hand a direct proof of (1) is possible. Let $C=sI-AA^*$ and let

$$M=\begin{pmatrix} 1 & v^*/(s-1) \\ 0 & I\end{pmatrix}.$$

Since $\det M=1$ we have $\det MC=\det C$. But

$$MC = \begin{pmatrix} s - \beta - \dfrac{v^* v}{s-1} & 0 \\ -v & (s-1)I \end{pmatrix},$$

from which (1) follows by direct computation.

Finally, we must recognize that $s = 1$ may be a root of $\det(sI - AA^*)$ of multiplicity greater than $n-2$. It is clear that $s = 1$ is a root of $(s-\beta)(s-1) - v^* v = 0$ iff $v^* v = 0$. Given that $v^* v = 0$, $s = 1$ is a simple root if $\beta \neq 1$ and a double root if $\beta = 1$. This latter case is the trivial one when $p(\lambda) = \lambda^n + p_n$ with $|p_n| = 1$.

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The Probability Integral Transformation: A Simple Proof

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One important transformation which is discussed in almost all introductory (or advanced) post calculus probability or statistic courses is the probability integral transformation $Y = F(X)$ where X is a random variable having continuous cumulative distribution function F. Of prime interest, particularly in a nonparametric setting, is the theorem: $Y = F(X)$ *is uniformly distributed over* $(0, 1)$. Most elementary textbooks (e.g., [**3**, p. 135], [**4**, p. 85], [**5**, p. 345]) prove this theorem only for special cases and some of the more advanced texts (e.g., [**1**, p. 37], [**2**, p. 146], [**6**, p. 18], [**7**, pp. 186–7]) seem to unduly complicate the general proof.

There seem to be two schools of thought on the proof of this transformation: one group would prefer an intuitive justification such as that in Fisz, whereas the other group would prefer a rigorous justification such as the one in Roussas. In this note I present an alternative for those choosing the route of mathematical rigor. This proof uses only the Intermediate Value Theorem (IVT) and the properties of a distribution function, so that the general proof can be understood readily by first semester calculus students.

Let $x \in (0, 1)$. Now F is continuous, so the IVT says that there must exist a z with $F(z) = x$. Since $x < 1$ we can find an ε (e.g., take $\varepsilon = (1-x)/2$) such that $x + \varepsilon/n < 1$ for each positive integer n. Again by the IVT there exists a z_n with $F(z_n) = x + \varepsilon/n$, each n. One can then use the monotonicity of F to observe that

$$\{X \leqq z\} \subseteq \{F(X) \leqq F(z)\} (= \{F(X) \leqq x\}) \subseteq \{F(X) < F(z) + \varepsilon/n\}$$
$$= \{F(X) < F(z_n)\} \subseteq \{X < z_n\} \subseteq \{X \leqq z_n\}.$$

Computing probabilities, we obtain

$$MC = \begin{pmatrix} s - \beta - \dfrac{v^* v}{s-1} & 0 \\ -v & (s-1)I \end{pmatrix},$$

from which (1) follows by direct computation.

Finally, we must recognize that $s = 1$ may be a root of $\det(sI - AA^*)$ of multiplicity greater than $n-2$. It is clear that $s = 1$ is a root of $(s-\beta)(s-1) - v^* v = 0$ iff $v^* v = 0$. Given that $v^* v = 0$, $s = 1$ is a simple root if $\beta \neq 1$ and a double root if $\beta = 1$. This latter case is the trivial one when $p(\lambda) = \lambda^n + p_n$ with $|p_n| = 1$.

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$$\begin{aligned} \{X \leqq z\} &\subseteq \{F(X) \leqq F(z)\} (= \{F(X) \leqq x\}) \subseteq \{F(X) < F(z) + \varepsilon/n\} \\ &= \{F(X) < F(z_n)\} \subseteq \{X < z_n\} \subseteq \{X \leqq z_n\}. \end{aligned}$$

Computing probabilities, we obtain

$$x = F(z) \leqq P(F(X) \leqq x) \leqq F(z_n) = x + \varepsilon/n.$$

Taking limits as n tends to infinity we find that $G(x) = P(F(X) \leqq x) = x$ for $x \in (0,1)$. Since G is right continuous, $G(0) = 0$; since $1 = G(1-) \leqq G(1) \leqq 1$, $G(1) = 1$.

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Non-Euclidean Domains: An Example

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Let Z denote the set of integers and, for each integer n that is positive and square free, let $Z[\sqrt{-n}] = \{a + b\sqrt{-n} \mid a, b \in Z\}$. It is well known [**1**] that the integral domains $Z[\sqrt{-n}]$ with $n > 2$ and $-n \equiv 2, 3 \pmod 4$ are non-Euclidean. In this note we give a new proof of this fact by applying a technique introduced by K. S. Williams in [**2**].

If D is an integral domain and $\tilde{D}$ is the set of all units of D together with zero, then $D = \tilde{D}$ if and only if D is a field. An element u of $D - \tilde{D}$ is called a **universal side divisor** of D if for each x in D there exists a member y of $\tilde{D}$ such that u divides $(x - y)$.

To begin our proof we take $D = Z[\sqrt{-n}]$ with $n > 2$ and $-n \equiv 2, 3 \pmod 4$; then $\tilde{D} = \{0, 1, -1\}$. Suppose (by way of contradiction) that D is Euclidean, and let ϕ be the usual norm on D defined by $\phi(a + b\sqrt{-n}) = a^2 + nb^2$. Consider the set $S = \{\phi(v) \mid v \in D - \tilde{D}\}$. By the well-ordering principle, S has a minimal element. If $\phi(v) \in S$, with $v = a + b\sqrt{-n}$, then $v \neq 1$, and the conditions on n imply that $\phi(v) \geqq 4$. Since $\phi(2) = 4$, the number 4 is the least element of S. Arguing on the lines of [**2**], we now show that 2 is a universal side divisor of D.

Let x be an arbitrary element of D. By the hypothesis that D is Euclidean, there exist $y, z \in D$ such that $x = 2y + z$ where $\phi(z) < \phi(2)$. Since $\phi(2)$ is the minimal element of S, this forces z to be a member of $\tilde{D}$. Thus 2 divides $(x - z)$, so 2 is a universal side divisor of D.

But now consider the algebraic integer $\sqrt{-n} \in D$. Since 2 is a universal side divisor of D, 2 must divide $\sqrt{-n} - y$ for some $y \in \tilde{D}$. But none of the three numbers $\sqrt{-n}/2$, $(\sqrt{-n} - 1)/2$ and $(\sqrt{-n} + 1)/2$ belongs to D, a contradiction. Thus D cannot be Euclidean, and the theorem is proved.

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The Ubiquitous e

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Texts on elementary matrix theory, e.g. [1], usually mention the following three methods for computation of the determinant of an n by n matrix $A = [a_{ij}]$:

(a) Direct computation using the definition via permutations;

(b) reduction by Gaussian elimination; and

(c) expansion by minors.

If we let a computation mean any addition, subtraction, multiplication or division (excluding those used in the determination of the signs), then it is easy to see that (a) requires $n!\,n-1$ computations. Many texts, e.g. [2], show that (b) requires $2n^3/3 - n^2/2 + 5n/6 - 1$ computations. These are used to compare the efficiency of (a) and (b), with the conclusion that (b) requires fewer computations for $n > 2$.

Here we wish to derive an explicit formula for the number of computations required by (c). Instead of just stating and proving the result, we will exhibit the process used to derive the formula so that the reader can share in the surprise of discovery.

Calculation of the determinant by expansion by the first row uses the equation $\det A = a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n}$, where the A_{ij} are the appropriate cofactors. If we denote by N_n the number of computations required by method (c), then calculation of a cofactor requires N_{n-1} computations. Once these are found, there are then n multiplications and $n-1$ additions to perform. This yields the recursion formula

$$N_n = nN_{n-1} + 2n - 1, \tag{1}$$

where $N_1 = 0$ (since no computations are required to find the determinant of a 1 by 1 matrix). N_n grows approximately at the same rate as $n!$ because $n!$ satisfies a similar recursion formula, namely, $n! = n(n-1)!$ So we will compare the rates of growth by computing the ratio $r = N_n/n!$ (see Table 1). This yields the somewhat surprising conjecture $\operatorname{Lim} N_n/n! = e = 2.7182818\ldots$.

To verify this we return to formula (1). By a straightforward use of induction on n, it is easy to show that

n	N_n	$n!$	$N_n/n!$
1	0	1	0
2	3	2	1.5
3	14	6	2.3
4	63	24	2.6
5	324	120	2.7
6	1,955	720	2.72
7	13,698	5,040	2.718
8	109,599	40,320	2.7182
9	986,408	362,880	2.71828
10	9,864,099	3,628,800	2.718281
11	108,505,110	39,916,800	2.7182818

A comparison between $n!$ and the number N_n of arithmetic computations required to compute a determinant of order n by the method of expansion by minors.

TABLE 1

$$N_n = n!\left(1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{(n-1)!}\right) - 1.$$

This can be expressed as

$$N_n = n!(e - E) - 1 = n!(e - E) + 1 - 2$$

where E is the remainder $\Sigma_{k=0}^{\infty} 1/(n+k)!$ in the Taylor expansion of e. Since for $n \geqq 1$,

$$1 < n!E = \sum_{k=0}^{\infty} \frac{n!}{(n+k)!} < \sum_{k=0}^{\infty} \frac{1}{(k+1)!} = e - 1 < 2,$$

it follows that

$$0 \leqq n!e - n!(e - E) - 1 < 1.$$

Now $n!(e - E)$ is an integer because it differs by 1 from N_n. Thus $[n!e]$, the greatest integer not exceeding $n!e$, is $n!(e - E) + 1$. Hence $N_n = [n!e] - 2$.

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Representations of Positive Integers as Sums of Arithmetic Progressions

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We define a positive arithmetic progression to be a sum of the form

$$m + (m + s) + \cdots + (m + ks)$$

where m and s are positive integers and $k \geqq 0$. We wish to examine the following question: In how many ways can a positive integer n be written as a positive arithmetic progression if the difference s is fixed? This is a generalization of the following questions which have been previously investigated. Given a positive integer, in how many ways can it be represented as a sum of consecutive integers [**2**]? In how many ways can a positive integer be written as a sum of consecutive odd integers [**3**]? Which positive integers have unique non-trivial representations as a sum of consecutive positive integers [**1**]? (A non-trivial representation is one in which $k \geqq 1$.)

We let $G(n, s)$ denote the number of representations of n as a positive arithmetic progression with difference s. We will prove that for s odd, $G(n, s)$ equals the number of odd divisors of n that are either less than some specific number σ_1 or greater than some other number σ_2. If s is even, we will show that $G(n, s)$ equals the number of divisors of n less than or equal to a third number σ_3. Our proof will include an algorithm for determining the actual representations.

The constants σ_1, σ_2, and σ_3 are given by the following formulas:

$$\sigma_1 = \frac{((s-1)^2 + 8sn)^{1/2} + (s-1)}{2s},$$

$$\sigma_2 = \frac{((s-1)^2 + 8sn)^{1/2} - (s-1)}{2},$$

$$\sigma_3 = \frac{((s-2)^2 + 8ns)^{1/2} + (s-2)}{2s}.$$

While these formulas may appear intimidating, they in fact arise quite naturally from solving certain quadratic inequalities by means of the quadratic formula.

The algorithm for determining the representations is the following. Suppose s is odd and d is a divisor of n less than σ_1. Then if we let the first term of the progression be

$$\frac{1}{2}\left(\frac{2n}{d} - (d-1)s\right)$$

we get a representation with d terms. If d divides n and d is greater than σ_2 the first term of the corresponding progression will be

$$\frac{1}{2}\left(d - \left(\frac{2n}{d} - 1\right)s\right)$$

and there will be $2n/d$ terms. Finally, if s is even and d is a divisor of n which is less than or equal to σ_3, we get a progression of d terms if we let the first term be

$$\frac{n}{d} - (d-1)\left(\frac{s}{2}\right).$$

Consider, for example, the case where $n = 45$ and the difference s is 2. In this case $\sigma_3 = \sqrt{45}$. The divisors of 45 which are not greater than $\sqrt{45}$ are 1, 3, and 5, so there are three representations whose first terms can be computed by applying the third formula given above: these first terms turn out to be 45, 13, and 5. The actual representations are $45 = 45$, $45 = 13 + 15 + 17$, and $45 = 5 + 7 + 9 + 11 + 13$.

Before proving our main result we will state some corollaries which are consequences of it. We use $\tau(n)$ to denote the number of divisors of n.

COROLLARY 1. *If s is odd, then the integers of the form 2^α, $\alpha \geqq 0$, have only the trivial representation; i.e., $2^\alpha = 2^\alpha$.*

COROLLARY 2. *If s is even, then all primes have only a trivial representation.*

COROLLARY 3. *If $s = 1$ and $n = 2^\alpha m$ where m is odd, then $G(n, 1) = \tau(m)$. (See* **[2]**, **[3]**, **[4]**.)

Proof. When $s = 1$, $\sigma_1 = \sigma_2 = \sqrt{2n}$. Since $\sqrt{2n}$ is never an odd integer, $G(n, 1)$ will be the number of all odd divisors of n, which is the same as the number of divisors of m.

COROLLARY 4. *The integers which have exactly one non-trivial representation as a sum of consecutive positive integers are those of the form $n = 2^\alpha p$ where p is an odd prime.*

Proof. This is equivalent to determining exactly when $G(n, 1) = 2$, i.e., when $\tau(m) = 2$. Thus $m = p$ and $n = 2^\alpha p$.

COROLLARY 5. *If $s = 2$, then $G(n, 2) = \tau(n)/2$ if n is not a perfect square and $G(n, 2) = (\tau(n) + 1)/2$ if n is a perfect square.*

Proof. When $s = 2$ we have $\sigma_3 = \sqrt{n}$. Thus $G(n, 2)$ equals $\tau(n)/2$ if $\sqrt{n}$ is not an integer, and $(\tau(n)+1)/2$ if n is a perfect square.

COROLLARY 6. *The integers n which have exactly one non-trivial representation as a positive arithmetic sequence with $s = 2$ are of the form $n = p^2$, $n = p^3$, or $n = p_1p_2$ where p, p_1 and p_2 are primes.*

Proof. We wish to find those integers that satisfy the relationship $G(n, 2) = 2$. Applying Corollary 5, we see that if n is not a perfect square, $G(n, 2) = 2$ if and only if $\tau(n)/2 = 2$ or $\tau(n) = 4$. This is the case if and only if $n = p^3$ or $n = p_1p_2$. If n is a perfect square then $G(n, 2) = 2$ if and only if $(\tau(n)+1)/2 = 2$ or $\tau(n) = 3$. This happens if and only if $n = p^2$.

We conclude with a formal statement and proof of our main result.

THEOREM. *The number $G(n, s)$ of representations of n as a positive arithmetic progression with difference s equals*

(i) *if s is odd, the number of odd divisors of n that are either less than*

$$\sigma_1 = \frac{((s-1)^2 + 8sn)^{1/2} + (s+1)}{2s}$$

or greater than

$$\sigma_2 = \frac{((s-1)^2 + 8sn)^{1/2} - (s-1)}{2};$$

(ii) *if s is even, the number of divisors of n less than or equal to*

$$\sigma_3 = \frac{((s-2)^2 + 8ns)^{1/2} + (s-2)}{2s}.$$

Moreover, the first term of a suitable arithmetic progression is given by

(a) $\frac{1}{2}(2n/d - (d-1)s)$ *if s and d are odd, $d \mid n$, $d < \sigma_1$;*
(b) $\frac{1}{2}(d - (2n/d - 1)s)$ *if s and d are odd, $d \mid n$, $d > \sigma_2$;*
(c) $n/d - (d-1)(s/2)$ *if s is even, $d \mid n$, $d \leqq \sigma_3$.*

Proof. First let us assume s is odd. Suppose n has a representation as a positive progression; i.e., $n = m + (m+s) + \cdots + (m+ks)$. Summing the right side we find that $2n = (2m+ks)(k+1)$. Notice that either $2m + ks$ or $k + 1$ is an odd divisor of n, but not both of them. Thus, for a given representation of n as a positive arithmetic progression we make a correspondence between the representation and $2m + ks$ or $k + 1$, whichever is odd. Since $m \geqq 1$, it follows that $k + 1 < 2m + k$. Since $2n = (2m + ks)(k + 1)$, it follows that $2n > (k + 1 + (s-1)k)(k+1)$. Solving this inequality for $k + 1$, by means of the quadratic formula, we get

$$k + 1 < \frac{((s-1)^2 + 8sn)^{1/2} + (s-1)}{2s} = \sigma_1$$

and

$$2m + ks = \frac{2n}{k+1} > \frac{2n}{\sigma_1} = \frac{((s-1)^2 + 8sn)^{1/2} - (s-1)}{2} = \sigma_2.$$

So the odd divisor that corresponds to the representation is either less than σ_1 or greater than σ_2. It is easy to show that $\sigma_2 \geqq \sigma_1$ so the divisor cannot be simultaneously less than σ_1 and greater than σ_2. A representation of n is completely determined by s and k, so when n and s are given, the value of $k + 1$ is different for each different representation. Thus, each representation of n corresponds to a distinct odd divisor of n.

If d is an odd divisor of n and $d < \sigma_1$, and if we let

$$m = \frac{1}{2}\left(\frac{2n}{d} - (d-1)s\right) \quad \text{and} \quad k = d - 1,$$

then we get

$$m + (m+s) + \cdots + (m + (d-1)s) = \frac{d}{2}[2m + (d-1)s] = \frac{d}{2}\left[\frac{2n}{d} - (d-1)s + (d-1)s\right] = n.$$

Similarly, if $d > \sigma_2$ and d is an odd divisor of n, we let

$$m = \frac{1}{2}\left(d - \left(\frac{2n}{d} - 1\right)s\right) \quad \text{and} \quad k = \left(\frac{2n}{d} - 1\right)$$

to obtain

$$m + (m+s) + \cdots + \left(m + \left(\frac{2n}{d} - 1\right)s\right) = \frac{1}{2}\left(\frac{2n}{d}\right)\left[2m + \left(\frac{2n}{d} - 1\right)s\right]$$
$$= \left(\frac{n}{d}\right)\left[\left(d - \left(\frac{2n}{d} - 1\right)s\right) + \left(\frac{2n}{d} - 1\right)s\right] = n.$$

So each odd divisor of n which is less than σ_1 or greater than σ_2 corresponds to a representation of n as a positive arithmetic progression. We conclude that there is a 1–1 correspondence between representations of n as a positive arithmetic progression and odd divisors of n which are either greater than σ_2 or less than σ_1; i.e., we have proved (i).

We now assume s is even in order to prove (ii). We have seen that $n = (k+1)(2m+ks)/2$; since $k + 1 \leqq k + m$, we obtain

$$n \leqq \frac{(2(k+1) + (s-2)k)}{2}(k+1).$$

Solving this inequality for $k+1$ we find that

$$k + 1 \leqq \frac{((s-2)^2 + 8ns)^{1/2} + (s-2)}{2s} = \sigma_3.$$

We now make a correspondence between representations of n and pairs of divisors of n, namely $k+1$ and $(2m+ks)/2$. This gives a distinct pair of divisors for each representation.

Now suppose that d divides n, and $d \leqq \sigma_3$. If we let

$$m = \frac{n}{d} - (d-1)\frac{s}{2} \quad \text{and} \quad k = d - 1$$

we get

$$m + (m+s) + \cdots + (m + (d-1)s) = \frac{d}{2}[2m + (d-1)s] = \frac{d}{2}\left[2\left(\frac{n}{d} - (d-1)\frac{s}{2}\right) + (d-1)s\right] = n.$$

We conclude that there is a 1–1 correspondence between representations and integers d which divide n and are less than or equal to σ_3. This completes the proof of (ii), and with it the theorem.

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Look, Ma, No Primes

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If n is a positive integer such that $\sqrt{n}$ is not an integer, then $\sqrt{n}$ is irrational. The usual proof of this depends on the theorem that a prime that divides a product must divide one of the factors. It might be amusing to give a proof that does not involve number theory.

Suppose, then, that $\sqrt{n} = x/y$, with x and y positive integers, x minimal. Since $\sqrt{n}$ is not an integer, by hypothesis, there is an integer k such that $k-1 < x/y < k$. Define $x' = (k-(x/y))x$ and $y' = (k-(x/y))y$. Clearly $x'/y' = x/y = \sqrt{n}$. Moreover, $x' = kx - (x^2/y) = kx - (ny^2/y) = kx - ny$ and $y' = ky - x$ are both integers, clearly positive. But $k - x/y < 1$, so $x' < x$. This contradicts the minimality of x and completes the proof.

The author would like to thank Professor R. Ayoub for pointing out that this proof is similar to one given by Dedekind in *Stetigkeit und irrationale Zahlen* which is translated in *Essays on Number*, Open Court, Chicago, 1901.

Multiplicative Magic Squares

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In the ordinary magic square, all rows and columns have the same sum. How about squares in which all rows and columns have the same *product*? We will call these p-magic squares. Here is an analysis of the 3×3 case.

It is easy to see that

$$A = A(a,b,c,d) = \begin{bmatrix} a & b & cd \\ d & c & ba \\ bc & da & 1 \end{bmatrix}$$

is a p-magic square. In fact, every nontrivial (without zero entries) p-magic 3×3 square is a multiple of some A. To see this, suppose B is a 3×3 nontrivial p-magic square. Let λ be the entry in the lower right hand corner of B. Factor out λ from all nine entries and write B as follows:

$$B = \lambda \begin{bmatrix} a & b & x \\ d & c & y \\ z & w & 1 \end{bmatrix}.$$

Since B is p-magic, it follows that $abx = xy$, so $y = ab$; $dcy = xy$, so $x = cd$; $adz = xy = abcd$, so $z = bc$; $bcw = abcd$, so $w = ad$. Consequently, $B = \lambda A(a,b,c,d)$. It is easy to show that the conditions $1 < a < b < c < d$, $ab \neq c$ or d, and $bc \neq d$ or ad are sufficient to guarantee that all nine entries in A are distinct. For example,

$$A(2,3,4,5) = \begin{bmatrix} 2 & 3 & 20 \\ 5 & 4 & 6 \\ 12 & 10 & 1 \end{bmatrix}$$

is a p-magic square with distinct entries.

We can pass from a p-magic square to an ordinary (additive) magic square by taking the logarithm of each entry (to any base), and vice versa by exponentiation. (Of course logarithms are not satisfactory if we want squares whose entries are all integers.) The reader may wish to prove in this manner, or directly, that the standard form A can be reformulated for ordinary 3×3 magic squares as

$$\begin{bmatrix} a & b & c+d \\ d & c & b+a \\ b+c & d+a & 0 \end{bmatrix}.$$

Another interesting project might be to investigate 4×4 and larger p-magic squares, or squares which are "magic" under an operation other than addition and multiplication.

Compact Subsets of the Sorgenfrey Line

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While the topology most commonly imposed on the set R of real numbers is the Euclidean topology $\mathcal{U}$ generated by the open intervals (a, b), the topology generated on R by the intervals of the form $[a, b)$ has received extensive study (see [**5**], pp. 24, 78, 88, 116, 182, 187 and [**3**], p. 59) and is among the stock counterexamples of general topology. Following Wilansky [**5**], we call this latter topology the **right hand open topology.** Since R with this topology is also frequently called the **Sorgenfrey Line** (after R. H. Sorgenfrey, who used it in [**4**] to answer a question raised in 1944 by J. Dieudonné in [**2**]), we will denote it by $\mathcal{S}$. We can see easily after that $\mathcal{U} \subset \mathcal{S}$ since for each $A \in \mathcal{U}$ and $x \in A$, there is an open interval (a, b) satisfying $x \in [x, b) \subset (a, b) \subset A$.

A standard result in elementary analysis is the Heine-Borel characterization of $\mathcal{U}$-compact subsets of R as precisely those subsets of R which are bounded and $\mathcal{U}$-closed (see [**5**], p. 83). In this note we provide two characterizations of the $\mathcal{S}$-compact subsets of R and then use these characterizations to prove that $\mathcal{S}$-compact subsets of R are countable.

We begin our search for characterizations by noting that since $\mathcal{U} \subset \mathcal{S}$, each $\mathcal{S}$-compact subset of R is $\mathcal{U}$-compact. Thus, an $\mathcal{S}$-compact subset of R must be bounded, $\mathcal{U}$-closed and consequently $\mathcal{S}$-closed. We then observe that a subset K of R which is bounded above and which contains a strictly increasing sequence $\{x_n\}$ cannot be $\mathcal{S}$-compact; for if $c = \sup\{x_n : n = 1, 2, 3, \ldots\}$, the collection $\mathcal{Q} = \{[x_n, x_{n+1}) : n = 1, 2, 3, \ldots\} \cup \{(-\infty, x_1), [c, \infty)\}$ of $\mathcal{S}$-open sets is a covering of K which contains no finite subcollection covering K since for each n, $x_n \in K$ and $[x_n, x_{n+1})$ is the only member of $\mathcal{Q}$ that has x_n as an element. Thus $\mathcal{S}$-compact subsets of R must be bounded, $\mathcal{S}$-closed, and devoid of strictly increasing sequences. Our first theorem characterizes the $\mathcal{S}$-compact subsets by proving the converse to this statement.

THEOREM 1. *A subset of R is $\mathcal{S}$-compact if and only if the subset is bounded, $\mathcal{S}$-closed and devoid of strictly increasing sequences.*

Proof. We need only prove that a bounded, $\mathcal{S}$-closed subset K of R which is devoid of increasing sequences must be $\mathcal{S}$-compact. Suppose $K \subset R$ is bounded, $\mathcal{S}$-closed, and suppose $\mathcal{Q}$ is an $\mathcal{S}$-open covering of K containing no finite subcollection that covers K. Then $K \neq \emptyset$ and if M is the union of any finite collection of elements of $\mathcal{Q}$, $K - M \neq \emptyset$. We now define a sequence of elements of K. Let $x_1 = \inf K$, and select $V_1 \in \mathcal{Q}$ with $x_1 \in V_1$. For $n > 1$ let $x_n = \inf(K - \bigcup_{k=1}^{n-1} V_k)$ and choose $V_n \in \mathcal{Q}$ with $x_n \in V_n$. This is possible since $K - \bigcup_{k=1}^{n-1} V_k \neq \emptyset$ for any n, and $x_n \in K$ for each n. Then by construction $x_{n+1} \geqq x_n$, $x_n \in V_n$ and $x_{n+1} \notin V_n$. Thus $\{x_n\}$ is a strictly increasing sequence in K, contrary to hypothesis. This contradiction completes the proof.

Another useful characterization of $\mathcal{U}$-compact subsets is the condition that each sequence in the subset has a subsequence which $\mathcal{U}$-converges to some point in the set (see [**1**], p. 45). It follows from this characterization that a subset K of R is $\mathcal{U}$-compact if and only if each strictly increasing sequence in K $\mathcal{U}$-converges to a point in K and each strictly decreasing sequence in K $\mathcal{U}$-converges to a point in K. We give a similar characterization of $\mathcal{S}$-compact subsets.

THEOREM 2. *A subset K of R is $\mathcal{S}$-compact if and only if each strictly decreasing sequence in K $\mathcal{S}$-converges to a point in K and K is devoid of strictly increasing sequences.*

Proof. If K is $\mathcal{S}$-compact, then K is devoid of strictly increasing sequences by Theorem 1. Also, K is bounded and $\mathcal{S}$-closed, so any strictly decreasing sequence $\{x_n\}$ in K converges to $\inf\{x_n : n = 1, 2, 3, \ldots\}$ which belongs to K.

We will prove the sufficiency of the conditions by showing that they force K to satisfy the conditions of Theorem 1. We begin by showing that K must be bounded. If K is not bounded above, choose $x_1 \in K \cap (1, \infty)$ and $x_n \in K \cap (\max\{n, x_{n-1}\}, \infty)$ for $n > 1$. Then $\{x_n\}$ would be a strictly increasing sequence in K, contrary to hypothesis. If K is not bounded below, choose $x_1 \in K \cap (-\infty, -1)$ and $x_n \in K \cap (-\infty, \min\{-n, x_{n-1}\})$ for $n > 1$. Then $\{x_n\}$ would be a strictly decreasing sequence in K which does not $\mathcal{S}$-converge to a point in K, thus yielding another contradiction. So K must be bounded. If K is not $\mathcal{S}$-closed, let x be in the $\mathcal{S}$-closure of K but not in K itself. Let $y_1 \in K \cap [x, x+1)$ and for $n > 1$, choose $y_n \in k \cap [x, \min\{x + 1/n, y_{n-1}\})$. Then $\{y_n\}$ would be, contrary to hypothesis, a strictly decreasing sequence in K which does not $\mathcal{S}$-converge to a point in K. Thus K is $\mathcal{S}$-closed and is consequently $\mathcal{S}$-compact by Theorem 1. This completes the proof.

COROLLARY 1. *Each $\mathcal{S}$-compact subset of R is countable.*

Proof. Let K be an uncountable subset of R. Then K is $\mathcal{S}$-Lindelöf (because every subset of R is $\mathcal{S}$-Lindelöf (see [**3**], p. 59)) and, because K is both uncountable and $\mathcal{S}$-Lindelöf, it must have an $\mathcal{S}$-condensation point — that is, a point with the property that every open set about it contains uncountably many elements of K. (Otherwise, we could cover K by open sets whose intersections with K are countable, and, by using the Lindelöf property to extract a countable subcover, we would have the uncountable K contained in a countable union of countable sets.) So let $x_1 \in K$ be an $\mathcal{S}$-condensation point of K, and for each $n > 1$, let x_n be an $\mathcal{S}$-condensation point of $(x_{n-1}, x_{n-1} + 1) \cap K$ which is in K. (This is possible since for each $n > 1$, $(x_{n-1}, x_{n-1} + 1) \cap K$ is uncountable and $\mathcal{S}$-Lindelöf.) Then $\{x_n\}$ is a strictly increasing sequence in K, so K is not $\mathcal{S}$-compact.

The following well-known result ([**5**], p. 88, exercise 205) follows easily from Corollary 1 since a nonempty element of $\mathcal{S}$ must be uncountable.

COROLLARY 2. *Each $\mathcal{S}$-compact subset of R has empty interior.*

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PROBLEMS

Dan Eustice, Editor
Leroy F. Meyers, Associate Editor
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Proposals

To be considered for publication, solutions should be mailed before June 1, 1977.

996. Suppose thumbtacks are used to tack congruent square sheets of paper to a large bulletin board subject to the following conditions:

(i) the sides of the sheet are parallel to the sides of the bound;

(ii) each sheet has exactly four thumbtacks, one in each corner; and

(iii) the sheets may overlap slightly so that one thumbtack could secure a corner of from one to four sheets.

(a) Find, in terms of n, the minimum number of thumbtacks required to tack n such sheets.

(b)* For a given n, find the number of distinct minimal arrangements.

(c)* Can the problem be generalized to hypercubes and hyperthumbtacks in three or more dimensions?

[*Richard A. Gibbs, Fort Lewis College.*]

997. Let P be a polynomial of degree n, $n \geqq 2$, with simple zeros $z_1, z_2, \ldots, z_n$. Let $\{g_k\}$ be the sequence of functions defined by $g_1 = 1/P'$, and $g_{k+1} = g_k'/P'$. Prove for all k that $\sum_{j=1}^{n} g_k(z_j) = 0$. [*John Lott, Student, Southwest High School, Kansas City, Missouri.*]

998. Characterize all triangles in which the triangle whose vertices are the feet of the internal angle bisectors is a right triangle. [*Hüseyin Demir, Middle East Technical University, Ankara, Turkey.*]

999. Let $\{a_i\}$ and $\{b_i\}$, $i = 1, 2, \ldots, k$, be natural numbers arranged in non-decreasing order. For which values of k is it true that $\sum_{i=1}^{k}(a_i!) = \sum_{i=1}^{k}(b_i!)$ implies $a_i = b_i$ for all i? What is the corresponding result if the two sequences are strictly increasing? [*Joseph Silverman, Brown University.*]

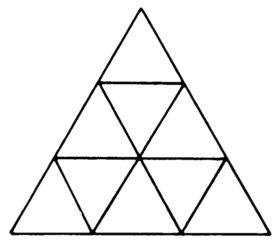
ASSISTANT EDITORS: Don Bonar, *Denison University*; William A. McWorter, Jr., *The Ohio State University. We invite readers to submit problems believed to be new. Proposals should be accompanied by solutions, when available, and by any information that will assist the editors. Solutions to published problems should be submitted on separate, signed sheets. An asterisk (*) will be placed by a problem to indicate that the proposer did not supply a solution. A problem submitted as a Quickie should be one that has an unexpected succinct solution. Readers desiring acknowledgement of their communications should include a self-addressed stamped card. Send all communications to this department to Dan Eustice, The Ohio State University, 231 W. 18th Ave., Columbus, Ohio 43210.*

1000. T denotes a cyclic permutation operator acting on the indices of a sequence $\{a_i\}$, that is, $T(a_1x_1 + a_2x_2 + \cdots + a_nx_n) = a_2x_1 + a_3x_2 + \cdots + a_1x_n$. If, for all i, $a_i \geqq 0$ and $x_i > 0$, show that

$$\left\{\sum_{i=1}^{n} \frac{a_i}{n}\right\}^n \geqq \prod_{i=1}^{n} T^i \left\{\frac{a_1x_1 + a_2x_2 + \cdots + a_nx_n}{x_1 + x_2 + \cdots + x_n}\right\} \geqq \prod_{i=1}^{n} a_i.$$

[*Murray S. Klamkin, University of Waterloo.*]

1001. In the accompanying figure, n, the length of the base is 3 units, and $f(n)$, the number of parallelograms, is 15. Find a formula for $f(n)$. (Cf. Problem 889, January, 1974, and Problem 975, March, 1976.) [*Edward T. H. Wang, Wilfrid Laurier University.*]



1002. a. For which values of n is it possible to find a permutation $[a_1, a_2, \ldots, a_n]$ of $[0, 1, \ldots, n-1]$ so that the partial sums $\sum_{i=1}^{k} a_i$, $k = 1, 2, \ldots, n$, when reduced modulo n, are also a permutation of $[0, 1, \ldots, n-1]$? [*Bernardo Recamán, University of Warwick.*]

b.* Find the number of permutations of $[0, 1, \ldots, n-1]$ for $n \leqq 12$ which solve part a. Can a general formula for the number of solutions be found? [*John Hoyt, Indiana University of Pennsylvania.*]

Quickies

Solutions to Quickies appear at the conclusion of the Problems section.

Q640. For positive integers n, find the values of

$$\text{(i)} \ \prod_{i=0}^{n-1} [n(n+1) - i(i+1)] \quad \text{and} \quad \text{(ii)} \ (n+1) \prod_{i=0}^{n-1} [n(n+2) - i(i+2)].$$

[*Peter A. Lindstrom, Genesee Community College.*]

Q641. If n and k are integers with $n > 2$ and $k \geqq 1$, show that n^k can be expressed as the sum of the squares of exactly n positive integers. [*Erwin Just and Norman Schaumberger, Bronx Community College.*]

Q642. Prove that the numbers 49, 4489, 444889, ..., obtained by inserting 48 into the middle of the preceding numbers, are all perfect squares. [*Steven R. Conrad, Benjamin N. Cardozo High School.*]

Solutions

Balancing Weights Again **September 1974**

914. If for any n of a given $n + 1$ integral weights, there exists a balance of them on a two pan balance where a fixed number of weights are placed on one pan and the remainder on the other pan, prove that the weights are equal. [*Murray S. Klamkin, University of Waterloo.*]

Editors' Comment. James A. Davis and Richard A. Gibbs point out that the published solution (September 1975) is incomplete. The argument that the balancing property of the initial weights w_i must be shared by $w_i/2$ or $(w_i - 1)/2$ fails just when all w_i are odd and the two pans contain unequally many weights. For example, $3+3+3=9$, but $1+1+1 \neq 4$. It should also be noted that the fixed number in the problem must be the same for every choice of n of the $n + 1$ weights. The necessity of this is seen for the set $\{1,1,1,1,3\}$ of weights.

II. *Solution*: We assume that the result holds if equally many weights are placed in the two pans, as proved in Solution I.

Let $S = \{w_1, w_2, \ldots, w_{n+1}\}$ be the given set of $n+1$ weights such that any n of them can be balanced with a suitable choice of k weights on one pan and $n - k$ weights on the other. Now consider a set $T = \{w_1, w_2, \ldots, w_{n+1}, w_1, w_2, \ldots, w_n\}$; that is, T consists of $2n + 1$ weights with two copies of the weights $w_1, w_2, \ldots, w_n$. Now if weight w_{n+1} is removed from T, the remaining weights balance with n weights on each pan. If weight w_i, $1 \leqq i \leqq n$, is removed from T, then T can be viewed as the union of $T_1 = \{w_1, w_2, \ldots, w_{i-1}, w_{i+1}, \ldots, w_{n+1}\}$ and $T_2 = \{w_1, w_2, \ldots, w_n\}$. Since both T_1 and T_2 can be divided into k and $n - k$ weights which balance, there is a balance of the weights of T with n weights on each pan. Thus, from Solution I, we conclude that all the weights of T, and hence S, are equal and that n must be even.

JAMES A. DAVIS
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Editors' Note. The problem generalizes to $n+1$ weights with real, positive values. A very nice solution (using linear algebra) to the problem with equally many weights on the two pans has been given by C. C. Clever and K. L. Yocom, this MAGAZINE, vol. 49, pp. 135–136.

A Subset of Integers Again **March 1975**

934. From the first kn positive integers, choose a subset, K, consisting of $(k-1)n+1$ distinct integers. Prove that at least one member of K is the sum of k members (not necessarily distinct) of K. [*Erwin Just, Bronx Community College.*]

Editors' Comment. Richard A. Gibbs and Edward T. H. Wang point out that, in general, the published solution (March 1976) is valid only if $n < k$, for if $n \geqq k$, then the removal of $2, 3, \ldots, n$ leaves none of the n pairs $(1,k), (2,2k), \ldots, (n, kn)$ complete, and hence no element of K is k times another.

On rechecking the solutions submitted, we found that all correct solutions, except that of Wedderburn (below) and the proposer, used mathematical induction.

II. *Solution*: Let m be the smallest number in the subset K. There are $n - m$ distinct numbers greater than m and not in K. We can assume $k \geqq 2$ since $k = 1$ is the trivial case. Consider the $k(n-m)+1$ identities

$$(k-1)m+1(m+i)=km+i, \qquad i=0,1,\ldots,k(n-m).$$

In each identity, the left hand side is the sum of k numbers all greater than or equal to m, and the right hand side is less than or equal to kn. Now there are at most $n-m$ values of i for which $m+i$ is not in K, and there are at most $n-m$ values of i for which $km+i$ is not in K, so that there are at most $2(n-m)$ values of i giving $m+i$ or $km+i$ (or both) not in K. Hence there are at least $k(n-m)+1-2(n-m)=(k-2)(n-m)+1$ values of i for which both $m+i$ and $km+i$ are in K. Since $k \geqq 2$, there is at least one number i where $km+i$ is in K and is the sum of k members of K, namely $k-1$ m's and one $m+i$.

G. Wedderburn
Tucker, Georgia

Maximal Subgroups **March 1975**

935. It is well known that the additive group Q of rational numbers has no maximal subgroup. Is this statement true for the multiplicative group Q^* of non-zero rational numbers? If the answer is no, then characterize all maximal subgroups of Q^*. [*Qazi Zameeruddin, K. M. College, Delhi* 7, *India.*]

Solution: (Adapted by the editors.) Let S be any basis (independent generating set) for the multiplicative group Q^+ of positive rational numbers. Then each of the following is easily seen to be a basis for a maximal subgroup of Q^*: (1) $(-A)\cup B$, where $A \subset S$, $-A=\{-a: a\in A\}$, and $B=S\backslash A$; (2) $(S\backslash\{a\})\cup\{a^p,-1\}$, where $a \in S$ and p is prime. In particular, Q^+ is a maximal subgroup of Q^*.

In fact, every maximal subgroup of Q^* must be one of these types. If M is any maximal subgroup of Q^*, then let $(-A)\cup B$ be a basis for M, where all elements of $A\cup B$ are positive.

(1) If $-1 \notin M$, then $A\cup B$ is a basis for Q^+, since otherwise adjoining -1 to M would yield a proper subgroup of Q^*.

(2) If $-1\in M$, then $A\cup B\cup\{-1\}$ is a basis for M, and $A\cup B$ is a basis for the maximal subgroup M^+ of Q^+. For a fixed a in $Q^+\backslash M^+$, let p be the smallest positive integer such that $a^p \in M^+$. (There must be such an integer, since otherwise $A\cup B\cup\{a^2\}$ would generate a group strictly between M^+ and Q^+.) Also, p must be prime, since otherwise $A\cup B\cup\{a^k\}$, where $k \mid p$ and $1<k<p$, would generate a group strictly between M^+ and Q^+. Now let $S=(A\cup B\cup\{a\})\backslash\{a^p\}$. Then S is a basis for Q^+, since it generates M^+ and more.

Francis J. Flanigan
San Diego State University

Also solved (existence of maximal subgroups) by J. C. Binz (Switzerland), Thomas E. Elsner, Donald C. Fuller, William Nuesslein, John D. O'Neill, Joseph Silverman, Y. H. Yin (Hong Kong), Ken Yocom, and the proposer.

Most attempted characterizations of the maximal subgroups assumed that every subgroup of Q^+ has a basis consisting of powers of primes. However, $\{2\cdot 3, 5, 7, \ldots\}$ generates a maximal subgroup of Q^+ not of this type. Some solvers of the first question found certain maximal subgroups of Q^+. All maximal subgroups of Q^+ can be shown to have a basis of the form $(S\backslash\{a\})\cup\{a^p\}$, where S is a basis for Q^+, $a\in S$, and p is prime.

Parallel Tangents **September 1975**

950. Show that there is a unique real number c such that for every differentiable function f on $[0,1]$ with $f(0)=0$ and $f(1)=1$, the equation $f'(x)=cx$ has a solution in $(0,1)$. [*Erwin Just, Bronx Community College.*]

Solution: Let $g(x) = x^2$. Then $c = 2$ is the only number such that $g'(x) = cx$ has a solution in (0, 1). Now let f be differentiable on [0, 1] with $f(0) = 0$ and $f(1) = 1$. Then $h = f - g$ is differentiable on [0, 1] with $h(0) = h(1) = 0$. Thus by Rolle's Theorem, $0 = h'(x) = f'(x) - 2x$ has a solution in (0, 1).

ANTHONY RICHOUX, Student
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Lafayette, Louisiana

Also solved by Mangho Ahuja, John T. Annulis, Gladwin Bartel, George Berzsenyi, M. T. Bird, Otha L. Britton, J. L. Brown, Jr., Arthur Charlesworth, Charles Clewer & Robert Lacher & Ken Yocom, Joseph D'Mello & V. Srinivas (India), Michael W. Ecker, Thomas E. Elsner, F. A. Ficken, William Fox, Richard Gibbs, Lee Hagglund, J. D. Hiscocks (England), Richard Johnsonbaugh, Henry S. Lieberman, John Oman, Aron Pinker, Bob Prielipp, Robert L. Raymond, Adam Riese, Daniel Rosenblum, Benjamin Schwartz, J. M. Stark, Alan H. Stein, Temple University Problem Solving Group, Julius Vogel, Edward T. H. Wang (Canada), William Wertman, Ernest Wilkins, Jr., and the proposer. Several solvers generalized the problem to pairs of functions with equal endpoints having parallel tangents. Three solvers used the stronger hypothesis of the derivative being integrable.

A Trace Condition **September 1975**

951. Let A be a square matrix, some scalar multiple of which differs from the identity matrix by a matrix of rank one. Give a simple necessary and sufficient condition that A be nonsingular, and find A^{-1} in this case. [*G. A. Heuer, Concordia College.*]

Solution: Let A be an $n \times n$ matrix. We are given that $kA = I + R$ where R is of rank one. Thus $R = ST$ where S and T are nonzero $n \times 1$ and $1 \times n$ matrices respectively, and $d = T \cdot S \neq 0$ is the trace of R. Now, R has an eigenvalue d of multiplicity 1 and an eigenvalue 0 of multiplicity $n - 1$ so that $I + R$ has eigenvalues $d + 1$ of multiplicity 1, and 1 of multiplicity $n - 1$. Therefore A is nonsingular if and only if $d \neq -1$. In this case, since $R^2 = dR$, it is easily verified that

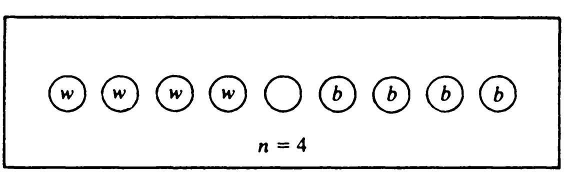
$$A^{-1} = kI - (k/(d+1))R.$$

RICHARD A. GIBBS
Fort Lewis College

Also solved by Joseph D'Mello & V. Srinivas (India), Thomas Elsner, Richard A. Groeneveld, John Z. Hearon, Edward T. H. Wang (Canada), and the proposer.

A Constant Bound **September 1975**

952. The object of a familiar puzzle is to interchange the positions of n white and n black pegs. One is allowed to jump pegs of opposite color, but never of the same color. A white (black) peg may move to the right (left) to an adjacent empty position.



Show that the transfer is always possible and establish a lower bound on the number of moves which is less than $2n(n+1)$.
[*F. D. Hammer, Stockton State College.*]

Solution: Each black peg moves left $n+1$ holes. Each white peg moves right the same number. The total number of spaces moved is $2n(n+1)$. But the pegs cannot pass each other without jumping, with white over black or vice versa. Each black peg must get to the left of n white pegs. For a white peg and black peg to pass each other requires one jump, which may be made by a peg of either color. Hence there are n jumps required for each of the n black pegs to pass or be passed by the white pegs. Each jump reduces the minimum number of required moves by one, so that the new lower bound is $n(n+2)$. This is, in fact, not a bound, but the *exact* value of the required number of moves. The description of a feasible strategy follows.

Alternately move k_i pegs of the same color in the same direction, i.e., move k_1 black pegs left, followed by k_2 white pegs right, etc. The strategy is then fully defined by the sequence k_i. Take

$$k_i = \begin{cases} i, & 1 \leqq i \leqq n \\ n, & i = n+1 \\ 2n - i + 2, & n+2 \leqq i \leqq 2n+1. \end{cases}$$

The number of moves is $\sum k_i = n(n+2)$, as previously derived.

BENJAMIN L. SCHWARTZ
McLean, Virginia

Also solved by George Berzsenyi, Thomas Elsner, Irwin K. Feinstein, Richard A. Gibbs, J. A. McDowell & R. J. Roedersheimer, G. Walther (Germany), William Wernick, Kenneth M. Wilke, M. Z. Williams, and the proposer. Gibbs observed that correct solutions to the intended problem establish upper bounds and not lower bounds.

Sum of Perfect Numbers **November 1975**

954. Show that any even perfect number greater than 28 can be represented as the sum of at least two perfect numbers. [*Richard L. Francis, Southeast Missouri State University.*]

Solution: We prove that any even perfect number $n > 28$ can be represented as linear combination of the perfect numbers 6 and 28. To do this we will use the following property of the natural numbers: If p and q are natural numbers such that $(p, q) = 1$, then every natural number greater than or equal to $pq - p - q + 1$ is obtained as a linear combination $mp + nq$, with m and n non-negative integers.

Now since $(3, 14) = 1$, every natural number greater than $3 \cdot 14 - 3 - 14 + 1 = 26$ can be represented as a linear combination of the form $3x + 14y$ with $x, y \geqq 0$. Multiplying by 2, we have that every even number greater than 52, in particular, an even perfect number $n > 28$, can be represented as a linear combination $6x + 28y$.

REINALDO E. GIUDICI
Universidad Simón Bolívar
Caracas, Venezuela

Also solved by Bern Problem Solving Group (Switzerland), David M. Burton, Richard Clark & Schuyler Grant, Clayton W. Dodge, Roger B. Eggleton, Thomas E. Elsner, R. S. Fisk & G. D. Peterson, Donald C. Fuller, Jeffrey A. Gates, Richard A. Gibbs, Leo Hämmerling (Germany), Earl E. Keese, M. S. Klamkin (Canada), Lew Kowarski, Graham Lord (Canada), T. E. Moore, Larry O. Olson, D. E. Penney, Bob Prielipp, John M. Samoylo, Joseph Silverman, Scott Smith, J. M. Stark, Julius Vogel, Edward T. H. Wang (Canada), Kenneth M. Wilke, and the proposer.

956. Let Q_m be the product of the first m primes: $Q_2 = 6$, $Q_3 = 30$, etc. Then, for $m \geqq 2$, $Q_m/2$ is the product of the first $m-1$ odd primes. Now $Q_2/2 = 2^1 + 1 = 2^2 - 1$, while $Q_3/2 = 2^4 - 1$. For $m > 3$, can $Q_m/2 = 2^j \pm 1$ for integer j? [*Arthur Marshall, Madison, Wisconsin.*]

Solution: If $m > 3$ and $Q_m/2 = 2^j + 1$ (with $j \geqq 0$) then $2^j \equiv -1 \pmod 7$. This is impossible, as the only residues of 2^j modulo 7 are 1, 2, and 4.

Now suppose that $m > 3$ and that $Q_m/2 = 2^j - 1$ for some integer $j \geqq 0$. Then $2^j \equiv 1 \pmod 5$ and $j \equiv 0 \pmod 4$. Also $2^j \equiv 1 \pmod 7$, so $j \equiv 0 \pmod 3$. Thus $j = 12k$, and

$$Q_m/2 = 2^{12k} - 1 = (2^{12} - 1)R$$

for some integer R. But $9 \mid (2^{12} - 1)$, so $9 \mid (Q_m/2)$. This too is impossible.

Thus if $m > 3$, it is impossible that $Q_m/2 = 2^j \pm 1$ for integral j.

D. E. Penney
University of Georgia

Also solved by Bern Problem Solving Group (Switzerland), Donald C. Fuller, Mark Kleiman, Larry O. Olson, Adam Riese, Kenneth M. Wilke, and the proposer.

Answers

Solutions to the Quickies which appear near the beginning of the Problems section.

Q640. To show that (i) has the value $(2n)!$, arrange the first $2n$ positive integers into two disjoint sets A and B, where $A = \{x_i : x_i = n - i\}$ and $B = \{y_i : y_i = n + i + 1\}$ for $i = 0, 1, \ldots, n-1$. Then

$$(2n)! = \prod_{i=0}^{n-1} [x_i y_i] = \prod_{i=0}^{n-1} [(n-i)(n+i+1)] = \prod_{i=0}^{n-1} [n(n+1) - i(i+1)].$$

By a similar procedure, the value of (ii) is $(2n+1)!$.

Q641. If $k = 2m$ with $m \geqq 1$, then

$$n^{2m} = [n^{m-1}(n-2)]^2 + 4n^{2m-2}(n-1) = [n^{m-1}(n-2)]^2 + \sum_{i=1}^{n-1} (2n^{m-1})^2.$$

If $k = 2m - 1$ with $m \geqq 1$, then

$$n^{2m-1} = n(n^{2m-2}) = \sum_{i=1}^{n} (n^{m-1})^2.$$

Q642. The nth number is

$$4 \cdot 10^n (1 + 10 + \cdots + 10^{n-1}) + 8(1 + 10 + \cdots + 10^{n-1}) + 1$$
$$= 4 \cdot 10^n \left(\frac{10^n - 1}{9}\right) + 8 \cdot \frac{10^n - 1}{9} + 1 = \frac{4}{9} \cdot 10^{2n} + \frac{4}{9} \cdot 10^n + \frac{1}{9} = \left(\frac{2 \cdot 10^n + 1}{3}\right)^2.$$

Editor's Comment. The two sequences 09, 1089, 110889, 11108889, ... and 81, 9801, 998001, 99980001, ... can be treated in a similar way.

NEWS & LETTERS

A FINITE CALCULUS ALTERNATIVE

If Samuel Goldberg's students had known numerical analysis (this *Magazine*, May 1976, pp. 130-131), they might have noticed a pattern in Table 1, that is:

$$X_{n,r} = S(n,r)\ [364]_{r-1}/365^{n-1} .$$

Here $S(n,r)$ is the Stirling number of the second kind and $[\ ]$ is the falling factorial. By definition, $E(X_{n,r})$ is just

$$\sum_{r=1}^{n} r\, S(n,r)\ [364]_{r-1}/365^{n-1} .$$

This expression can be simplified by using finite calculus. If Δ is the difference operator then

$$\Delta[365]_r = r[364]_{r-1} .$$

Also from finite calculus we have that

$$\sum_{r=1}^{n} S(n,r)[365]_r + \sum_{r=1}^{n} S(n,r)[364]_r$$

is precisely $365^n + 364^n$. Substituting these results into the expression for $E(X_{n,r})$ yields

$$E(X_{n,r}) = 365 - 364^n/365^{n-1} .$$

This is the same result that Samuel Goldberg gets, but the solution did not need a "gimmick" before a pattern arose.

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CLOSING THE GAP

Professor D. Grant's assertion (this *Magazine*, Sept. 1975, p. 218) that the normalizer $N_G(A)$ of a subgroup A in a group G is given by $N_G(A) = \{x:C_a(x)A = xA$ for all $a \in A\}$ where $C_a(x) = axa^{-1}$ is incorrect. Because $x^{-1}Ax$ may be a proper subset of A having the same cardinality as A, $x^{-1}Ax \subseteq A$ need not imply that $x^{-1}Ax = A$, as Professor Grant tacitly assumes. Let A be the set of 2 by 2 matrices of the form $\left(\begin{smallmatrix}1 & 2\\ 0 & 1\end{smallmatrix}\right)$ with Z an integer. A forms a subgroup of the group of ax non-singular 2 x 2 real matrices under matrix multiplication. For $x = \left(\begin{smallmatrix}1 & 0\\ 0 & 2\end{smallmatrix}\right)$, $x^{-1}Ax \neq A$ and the given characterization of $N_G(A)$ fails. However the easily checked observation that $N_G(A)$ is the intersection of $\{x:C_a(x)A = xA$ for all $a \in A\}$ with $\{x:AC_a(x) = Ax$ for all $a \in A\}$ does yield an easy proof that $N_G(A)$ is closed when A is a closed subgroup of a Hausdorff group G.

W. Hugh Haynsworth
D.P. Yeager
College of Charleston

AND CONVERSELY

In a recent article (this *Magazine*, Jan. 1976, pp. 32-34), Jungck proved that a continuous self-mapping f on a compact metric space (X,P) satisfying $P(f^2(x),f^2(y)) < P(f(x),f(y))$ for all $x,y \in X$ with $f(x) \neq f(y)$, has a unique fixed point. He remarked that Edelstein's result (*J. London Math. Soc.*, 37 (1962) 74-79) is a special case of his. We can view the situation the other way round. For f as above set $X_0 = f(x)$. Clearly X_0 is compact, f is a self-mapping on X_0, and on X_0 f satisfies $P(f(x),f(y)) < P(x,y)$ for all $x,y \in X_0$ with $x \neq y$; thus there is a unique fixed point of f by Edelstein's theorem.

We offer the following generalization of Edelstein's result. We call a self-mapping f on a metric space X contractive if for all closed bounded subsets $B \subseteq X$ with $f(B) \subseteq B$ and $\delta(B) > 0$ we have $\delta(f(B)) < \delta(B)$, where $\delta(A)$ denotes the diameter of A. Then a contractive continuous self-mapping on a compact metric space X has a unique fixed point. Consider $U = \{U_\alpha \subseteq X: U_\alpha$ is closed and $f(U_\alpha) \subseteq U_\alpha\}$. Apply Zorn's lemma to obtain a minimal element $A \in U$. Since $f^2(A) \subseteq f(A)$ and $f(A)$ is closed, minimality of A yields $f(A) = A$. The contractiveness of f now implies that A is a single point. Uniqueness follows easily.

Let f be a continuous self-mapping on a compact metric space (X,P) satisfying the condition: there are nonnegative real numbers a_i, $i = 1,2,3,4,5$ with $\sum_{i=1}^{5} a_i \leq 1$ such that for all $x,y \in X$,

$$P(f(x),f(y)) < [a_1P(x,y) + a_2P(x,f)x)) + a_3P(y,f(y)) + a_4P(x,f(y)) + a_5P(y,f(x))].$$

Then f is contractive in our sense and moreover, when $a_1 = 1$, $a_2 = a_3 = a_4 = a_5 = 0$, f satisfies Edelstein's criterion. Thus the above result includes Edelstein's result.

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THE CONWAY STONES

In reference to "The Conway Stones: What the Original Hebrew May Have Been" (this *Magazine*, Sept. 1976, pp. 207-210) I am able to read the Hebrew script after a fashion and I can tell you that it mimics and is extremely disrespectful of what many consider to be sacred scripture. As editors of *Mathematics Magazine* you could not know this but the second author, Moshe Yavne, with his "native command of Hebrew" knows this.

This article does not belong in *Mathematics Magazine*. There are ways of teaching mathematics without resorting to this trash.

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Editors Note: We are sorry if some readers may have taken offense at the Hebrew text in this article. The entire article was read before publication both by a mathematician literate in modern Hebrew and by a biblical scholar knowledgeable in the sacred texts. Neither of these referees suggested that the article might offend readers, and both recommended that the article be published as a neat piece of light-weight humor. We apologize to readers who found it otherwise.

FIXED POINT REFERENCE

The result reported in B. Fisher's "A Fixed Point Theorem" (this *Magazine*, September 1975, pp. 223-225) as well as some more general results were proved in 1972 by T. Zamfirescu, "Fixed Point Theorems in Metric Spaces", *Arch. Math.*, 23 (1972) 292-298.

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TWO MATRIX ALTERNATIVES

May I make two comments on Roger Chalkley's paper "Matrices Derived from Finite Abelian Groups" (this *Magazine*, May 1976, pp. 121-129)? First, Chalkley discusses the diagonalization of the regular matrix representation of an element of the group algebra $F[G]$, where G is a finite abelian group; these results seem to be a restatement of the properties of the multidimensional discrete Fourier transform.

Second, using the transitivity of the determinant (M.H. Ingraham, *Bull. Amer. Math. Soc.*, 43 (1937) 579-580), one can elegantly prove Theorem 3 of the paper by induction on the number of cyclic groups constituting G. For example, suppose $G = C_3 \times H$, where C_3 is the cyclic group of order 3. With respect to an appropriate basis the regular matrix representation of an element of $F[G]$ has the form

$$A = \begin{bmatrix} A_0 & A_2 & A_1 \\ A_1 & A_0 & A_2 \\ A_2 & A_1 & A_0 \end{bmatrix}$$

where A_0, A_1, and A_2 are regular matrix representations of certain elements of $F[H]$. Letting w denote a primitive cube root of unity and I denote the identity matrix of appropriate size, we have $\det(\lambda I - A) = \det[(\lambda I - (A_0 + A_1 + A_2))(\lambda I - (A_0 + wA_1 + w^2A_2))(\lambda I - (A_0 + w^2A_1 + wA_2))]$.

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FULL CIRCLE

I would like to offer the following observations concerning Dan Pedoe's article, "The Most "Elementary" Theorem of Euclidian Geometry" (this *Magazine*, January 1976, pp. 40-42). For a complet quadrangle $ABCD$, there are <u>two</u> somewhat similar, but unrelated theorems:

I. If $AB+DC=BC+AD$, then $AE+CF=AF+CE$.
II. If $AB+BC=CD+DA$, then $AE+EC=AF+FC$.

The first of these can be proved by noting that the condition $AB+DC=BC+AD$ implies that quadrangle $ABCD$ has an inscribed circle. This circle is also inscribed in quadrangle $AFCE$, hence $AE+CF=AF+CE$.

The second of the two is the theorem attributed by Pedoe to M.L. Urquhart. If $AB+BC=CD+DA$, the quadrangle $ABCD$ does <u>not</u> have an inscribed circle. Theorem II can be proven, without the use of circles, by computing DF, FC, CE and BE either by trigonometric methods or by applying Stewart's theorem to the triangles ACF, ABF, ACE and ADE. Both methods lead to excessive computations.

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California 91030

ERRATUM

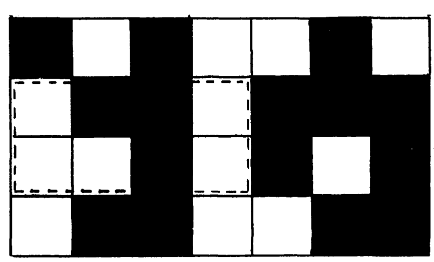
The eagle eyes of Professors W. Sit and C. Sit (both of CCNY) uncovered a slip in my paper "When is -1 a power of 2?" (this *Magazine*, November 1975, pp. 284-286). In Theorem 4, all conditions should refer to p instead of to d, where d is a power of the odd prime p. In fact, the proof only shows that much. The Sits have also given a more complete necessary and sufficient condition. I like to thank them warmly.

Man-Keung Siu
U. of Hong Kong

U.S.A. OLYMPIAD SOLUTIONS

The fifth U.S.A. Mathematical Olympiad took place on May 4, 1976, and the problems were published that same month in this column. The following sketches of solutions were adapted by Loren Larson from Samuel Greitzer's pamphlet "Mathematical Contests for 1976."

1. (a) Suppose that each square of a 4 x 7 chessboard, as shown below, is colored either black or white. Prove that with any such coloring, the board must contain a rectangle (formed by the horizontal and vertical lines of the board), such as the one outlined in the figure, whose four distinct corner squares are all of the same color.
(b) Exhibit a black-white coloring of a 4 x 6 board in which the four corner squares of every such rectangle as described above are not the same color.



Sol. (a). Such a rectangle must exist even in a 3 x 7 chessboard, as may be seen by examining the possible colorations of the columns and noting that at least 11 of the 21 squares must have the same color.

(b). Place two whites and two blacks in each column in the 4!/2!2!=6 possible different ways.

2. If A and B are fixed points on a given circle and XY is a variable diameter of the same circle, determine (with proof) the locus of the point of intersection of lines AX and BY. You may assume that AB is not a diameter.

Sol. Let P and P' denote respectively the intersection of AX and AY when this intersection lies outside or inside the circle. In both cases $\angle YAX$ is a right angle and $\angle AYB$ is constant. Hence, $\angle APB$, the complement of $\angle AYB$, is constant. This implies that P lies on the circle which is the locus of points making a constant angle with the constant base AB of ΔAPB. Also $\angle AP'Y$ is the complement of $\angle AYB$ and is therefore constant so again P' lies on a circle. Since $\angle P$ and $\angle AP'B$ are supplementary, P and P' lie on the same circle. The radius of the circle can be found from the law of sines to be $R = AB/2 \sin P$.

3. Determine (with proof) all integral solutions of $a^2 + b^2 + c^2 = a^2b^2$.

Sol. Analysis of the equation modulo 4 establishes that each of the integers must be even. This leads to an infinite descent, which can be resolved only if $a = b = c = 0$.

4. If the sum of the lengths of the six edges of a trirectangular tetrahedron $PABC$ (i.e., $\angle APB = \angle BPC = \angle CPA = 90°$) is S, determine (with proof) its maximum volume.

Sol. Let $AP = a$, $BP = b$, $CP = c$. Then

$$\begin{aligned} S &= a + b + c \\ &\quad + \sqrt{a^2+b^2} + \sqrt{b^2+c^2} + \sqrt{c^2+a^2} \\ &\geq (a+b+c)^2 + (\sqrt{2ab} + \sqrt{2bc} + \sqrt{2ac}) \\ &\geq 3(abc)^{1/3} + 3(\sqrt{2ab}\ \sqrt{2bc}\ \sqrt{2ac})^{1/3} \\ &= 3(1+\sqrt{2})(abc)^{1/3} \end{aligned}$$

with equality if and only if $a = b = c$. Since $V = abc/6$, we can substitute and get $V = S^3/162(1+\sqrt{2})^3$ or $V_{max} = (5\sqrt{2}-7)S^3/162$.

5. If $P(x)$, $Q(x)$, $R(x)$, and $S(x)$ are all polynomials such that

$$P(x^5) + xQ(x^5) + x^2R(x^5) = (x^4 + x^3 + x^2 + x + 1)S(x) ,$$

prove that $x - 1$ is a factor of $P(x)$.

Sol. Let $Eq(x)$ denote the given equation and let ω denote a fifth root of unity. Then

$$\begin{aligned} &Eq(\omega) + Eq(\omega^2) + Eq(\omega^3) + Eq(\omega^4) \\ &- [\omega Eq(\omega) + \omega^2 Eq(\omega^2) + \\ &\quad \omega^2 Eq(\omega^3) + \omega^4 Eq(\omega^4)] \end{aligned}$$

is the equation $5P(1) = 0$, which implies that $x - 1$ is a factor of $P(x)$.

INTERNATIONAL OLYMPIAD SOLUTIONS

The September issue of Mathematics Magazine contained the problems from the 18th International Mathematical Olympiad, which took place in Austria in July 1976. Here are sketches of solutions to these olympian problems for readers who wish aid or confirmation.

1. In a plane convex quadrilateral of area 32 cm^2 the sum of the lengths of two opposite sides and one diagonal is equal to 16 cm. Determine all possible lengths of the other diagonal.

Sol. Let $ABCD$ be the quadrilateral with sides $AB = a$, $CD = c$ and diagonal $AC = d$ such that $a + c + d = 16$. Ignoring the area, we see that the second diagonal can vary from zero to 16, and we investigate whether such a quadrilateral can have an area as large as 32. The maximum area will be obtained when AB and CD are perpendicular to AC, and in this case the area is $(a+c)d/2$ which is a maximum when $a + c = d = 8$, and in this case the area is 32. Therefore, the desired quadrilateral has maximum area, and we find the second diagonal must equal $8\sqrt{2}$.

2. Let $P_1(x) = x^2 - 2$ and $P_j(x) = P_1(P_{j-1}(x))$ for $j = 2, 3, \ldots$. Show that for any positive integer n, the roots of the equation $P_n(x) = x$ are all real and distinct.

Sol. We can easily show by induction that for $|x| > 2$, $P_n(x) > 2 > x$. Assuming the roots have the form $x = 2 \cos t$, we show that $P_n(2 \cos t) = 2 \cos 2^n t$. However $2 \cos 2^n t = 2 \cos t$ has the 2^n different solutions

$t = 2\pi m/(2^n-1)$, $m = 0,1,2,\ldots,2^{n-1} - 1$

and

$t = 2\pi m/(2^n+1)$, $m = 1,2,3,\ldots,2^{n-1}$.

3. A rectangular box can be filled completely with unit cubes. If one places as many cubes as possible, each with volume 2, in the box so that their edges are parallel to the edges of the box, one can fill exactly 40% of the box. Determine the dimensions of all such boxes.

Sol. Suppose $a_1 \leq a_2 \leq a_3$ are the dimensions of the box and $b_i \equiv [a_i/\sqrt[3]{2}]$. Then $(2/5)a_1a_2a_3 = 2b_1b_2b_3$ or $(a_1a_2a_3)/(b_1b_2b_3) = 5$. This key condition on the a_i/b_i together with a short table of values a_i/b_i for $a_i = 2,3,\ldots,10$ shows that $a_1 = 2$, and $2 < a_2 < 6$. Then it is easy to check that $a_2 = 3$ yields $a_3 = 5$; $a_2 = 4$ implies $a_3/b_3 = 15/8$ which is impossible; and $a_2 = 5$ determines $a_3 = 6$. Therefore the box is either $2 \times 3 \times 5$ or $2 \times 5 \times 6$.

4. Determine, with proof, the largest number which is the product of positive integers whose sum is 1976.

Sol. The largest product has the form 2^x3^y since a summand $a \geq 4$ can be replaced by the two terms $a - 2$ and 2, and obviously the largest product will have no factors of 1. Now, since $2^3 < 3^2$, we can say that in the largest product $x = 0$, 1 or 2. Thus the largest product of $1976 = 3 \times 658 + 2$ is 2×3^{658}.

5. Consider the system of p equations in q unknowns, where $q = 2p$,

$$\begin{aligned} a_{11}x_1 + \ldots + a_{1q}x_q &= 0 \\ a_{21}x_1 + \ldots + a_{2q}x_q &= 0 \\ &\ldots \\ a_{p1}x_1 + \ldots + a_{pq}x_q &= 0 \end{aligned}$$

with every coefficient a_{ij} a member of the set $\{-1,0,+1\}$. Prove that there exists a solution $(x_1,\ldots,x_q)$ of the system such that

(a) all x_j $(j = 1,\ldots,q)$ are integers;

(b) there is at least one value of j for which $x_j \neq 0$;

(c) $|x_j| \leq q$ $(j = 1,\ldots,q)$.

Sol. Let A be the matrix of coefficients and X the $q \times 1$ column vector whose entry in position i is x_i. There are $(2p+1)^q$ possible vectors X such that $|x_i| \leq p$. For such vectors X,

$$|a_{i1}x_1 + \ldots + a_{iq}x_q| \leq pq$$

which means that there are at most $(2pq+1)^p$ possible images AX. Since $(2p+1)^q = (2p+1)^{2p} = (4p^2+4p+1)^p > (4p^2+1)^p = (2pq+1)^p$, there must be two vectors X_1 and X_2 such that $AX_1 = AX_2$. The components of (X_1-X_2) satisfy all the requirements of the problem.

6. A sequence $\{u_n\}$ is defined by

$$u_0 = 2,\ u_1 = 5/2,$$

$$u_{n+1} = u_n(u_{n-1}^2 - 2) - u_1$$

for $n = 1,2,\ldots$. Prove that for positive integral n

$$[u_n] = 2^{(2^n-(-1)^n)/3}$$

where $[x]$ denotes the greatest integer less than or equal to x.

Sol. Inspection of the first few terms of the sequence leads one to suspect that

$$u_n = 2^{f(n)} + 2^{-f(n)}$$

for some function f. An attempt at proving this conjecture by induction makes $f(k+1) = f(k) + 2f(k-1)$ a necessary condition. The associated difference equation has the solution $f(n) = (2^n-(-1)^n)/3$, and direct substitution confirms that this f makes the suspected form of u_n correct. Since $f(n)$ is always an integer, $2^{-f(n)}$ is a proper fraction and the problem is solved.

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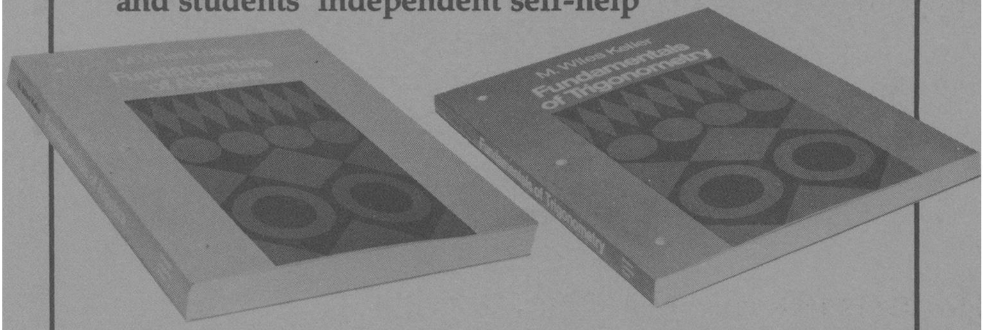

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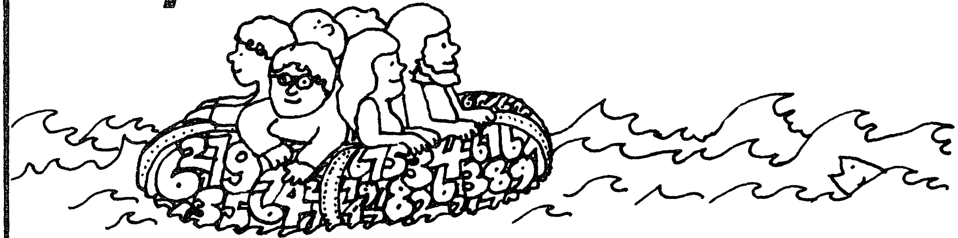

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